E435: spin 1 pure polarization states

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The problem:

For spin 1/2 we have found that $|\langle\vec{e_1}|\vec{e_2}\rangle|^2 = \frac{1}{2}(1+\vec{e_1}\cdot\vec{e_2})$.
Find the expressions that generalize the result for the case of spin 1.

Distinguish 3 cases:

(1) Both e_1 and e_2 have circular polarization
(2) Both e_1 and e_2 have linear polarization
(3) One has linear and the second circular polarization

The solution:

The standard basis for the states of spin 1 is defined as follow :

$$|m=1\rangle = |\Uparrow\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |\overrightarrow{e_z}\rangle$$

$$|m=0\rangle = |\Updownarrow\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |\overline{e_z}\rangle$$

$$|m=-1\rangle = |\Downarrow\rangle \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = |-\overrightarrow{e_z}\rangle$$

The first and the last states represent circular polarizations while the middle state describes linear polarization

(1) A pure spin 1 circular polarization is defined by :
Turning an $\Uparrow$ state by angle θ around y axis then turning by ϕ angle around z axis.
Hence in order to find the inner product of two spin 1 pure circular polarization states one shall mark , without loss of generality , $e_1 = \overrightarrow{e_z}$
and get :

$$|\overrightarrow{e_{\theta,\varphi}}\rangle = U(\varphi\overrightarrow{e_z})U(\theta\overrightarrow{e_y})\,|\Uparrow\rangle = \begin{pmatrix} \frac{1}{2}\,(1+\cos\theta)\,e^{-i\varphi} \\ \frac{1}{\sqrt{2}}\sin\theta \\ \frac{1}{2}\,(1-\cos\theta)\,e^{i\varphi} \end{pmatrix}$$

$$\langle e_1 \mid e_2\rangle = \langle\Uparrow \mid \overrightarrow{e_{\theta,\varphi}}\rangle = \langle\Uparrow \mid U(\varphi\overrightarrow{e_z})U(\theta\overrightarrow{e_y})\,|\Uparrow\rangle = \frac{1}{2}\,(1+\cos\theta)\,e^{-i\varphi}$$

$$|\langle e_1 \mid e_2\rangle|^2 = \frac{1}{4}\,(1+\cos\theta)^2$$

We can easly see that the result can be expressed in terms of $e_1 \cdot e_2$
by the expression :

$$|\langle e_1 \mid e_2\rangle|^2 = \frac{1}{4}\,(1+(e_1\cdot e_2))^2$$

(2) A pure spin 1 linear polarization is defined by :
Turning an $\Updownarrow$ state by angle θ around y axis then turning by ϕ angle around z axis.
Hence in order to find the inner product of two spin 1 pure circular polarization states one shall mark ,without loss of generality , $e_1 = \overline{e_z}$
and get :

$$|\overline{e_{\theta,\varphi}}\rangle = U(\varphi\overrightarrow{e_z})U(\theta\overrightarrow{e_y})\,|\Updownarrow\rangle = \begin{pmatrix} -\frac{1}{\sqrt{2}}\sin\theta e^{-i\varphi} \\ \cos\theta \\ \frac{1}{\sqrt{2}}\sin\theta e^{i\varphi} \end{pmatrix}$$

$$\langle e_1 \mid e_2 \rangle = \langle \Updownarrow \mid \overline{e_{\theta,\varphi}} \rangle = \langle \Updownarrow \mid U(\varphi\overrightarrow{e_z})U(\theta\overrightarrow{e_y})\,|\Updownarrow\rangle = \cos\theta$$

$$|\langle e_1 \mid e_2 \rangle|^2 = \cos^2\theta$$

We can easly see that the result can be expressed in terms of $e_1 \cdot e_2$
by the expression :

$$|\langle e_1 \mid e_2 \rangle|^2 = (e_1 \cdot e_2)^2$$

(3) A case in which e_1 have linear polarization and e_2 have circular polarization .
one shall mark ,without loss of generality , either $e_1 = \overline{e_z}$ and get :

$$|\overrightarrow{e_{\theta,\varphi}}\rangle = U(\varphi\overrightarrow{e_z})U(\theta\overrightarrow{e_y})\,|\Uparrow\rangle = \begin{pmatrix} \frac{1}{2}\left(1+\cos\theta\right)e^{-i\varphi} \\ \frac{1}{\sqrt{2}}\sin\theta \\ \frac{1}{2}\left(1-\cos\theta\right)e^{i\varphi} \end{pmatrix}$$

$$\langle e_1 \mid e_2 \rangle = \langle \Updownarrow \mid \overrightarrow{e_{\theta,\varphi}} \rangle = \langle \Updownarrow |\, U\left(\varphi\overrightarrow{e_z}\right) U\left(\theta\overrightarrow{e_y}\right) |\Uparrow\rangle = \frac{1}{\sqrt{2}}\sin\theta$$

or $e_1 = \overrightarrow{e_z}$,
and get :

$$|\overline{e_{\theta,\varphi}}\rangle = U(\varphi\overrightarrow{e_z})U(\theta\overrightarrow{e_y})\,|\Updownarrow\rangle = \begin{pmatrix} -\frac{1}{\sqrt{2}}\sin\theta e^{-i\varphi} \\ \cos\theta \\ \frac{1}{\sqrt{2}}\sin\theta e^{i\varphi} \end{pmatrix}$$

$$\langle e_1 \mid e_2 \rangle = \langle \Uparrow \mid \overline{e_{\theta,\varphi}} \rangle = \langle \Uparrow |\, U\left(\varphi\overrightarrow{e_z}\right) U\left(\theta\overrightarrow{e_y}\right) |\Updownarrow\rangle = -\frac{1}{\sqrt{2}}\sin\theta e^{-i\varphi}$$

we can see that the result is consistent with our assumptiones since for bouth choices we get :

$$|\langle e_1 \mid e_2 \rangle|^2 = \frac{1}{2}\sin^2\theta$$

and the result in terms of $e_1 \cdot e_2$
is given by :

$$|\langle e_1 \mid e_2 \rangle|^2 = \frac{1}{2}\left(1 - (e_1 \cdot e_2)^2\right)$$