# E432: Pure polarization states of spin $\frac{1}{2}$

**Submitted by: Zvika Oscar**

**The problem:**

A pure spin $\frac{1}{2}$ polarization is defined by: Turning an up state by angle $\theta$ around y axis then turning by $\phi$ angle around z axis.

(1) Write the resultant state as a superposition of up $|\uparrow\rangle$ and down $|\downarrow\rangle$ states.
(2) Show explicitly it is an eigenstate of the rotation generator around $n_{\theta,\phi}$ axis.
(3) Without further calculation conclude the following result:

$$|\langle\vec{e_1}|\vec{e_2}\rangle|^2 = \frac{1}{2}(1+\vec{e_1}\cdot\vec{e_2})$$

where $\vec{e_1}, \vec{e_2}$ are any two polarization directions

**The solution:**

(1) The representation of pure spin $\frac{1}{2}$ polarization as a superposition of up $|\uparrow\rangle$ and down $|\downarrow\rangle$ states is given by:

$$|\vec{e_{\theta,\phi}}\rangle = R_z(\phi)R_y(\theta)|\uparrow\rangle = \mathrm{e}^{-i\phi S_z}\mathrm{e}^{-i\theta S_y}|\uparrow\rangle = \mathrm{e}^{-i\phi/2}\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \mathrm{e}^{i\phi/2}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle$$

The two rotation operators are represented by:

$$R_y(\theta) = \mathrm{e}^{-i\theta\cdot\frac{1}{2}\sigma_y} = \cos\frac{\theta}{2} - i\sigma_y\sin\frac{\theta}{2} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_z(\phi) = \mathrm{e}^{-i\phi\cdot\frac{1}{2}\sigma_z} = \cos\frac{\phi}{2} - i\sigma_z\sin\frac{\phi}{2} = \begin{pmatrix} \mathrm{e}^{-i\frac{\phi}{2}} & 0 \\ 0 & \mathrm{e}^{i\frac{\phi}{2}} \end{pmatrix}$$

(2) First we find $S_n$

$$\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$S_n = \vec{n}\cdot\vec{S} = \vec{n}\cdot\frac{1}{2}\vec{\sigma} = \frac{1}{2}\begin{pmatrix} \cos\theta & \mathrm{e}^{-i\phi}\sin\theta \\ \mathrm{e}^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}$$

Then we use $S_n$ operator on the pure spin $\frac{1}{2}$ polarization from (1)

$$S_n|\vec{e_{\theta,\phi}}\rangle = \frac{1}{2}\begin{pmatrix} \cos\theta & \mathrm{e}^{-i\phi}\sin\theta \\ \mathrm{e}^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-i\phi/2}\cos\left(\frac{\theta}{2}\right) \\ \mathrm{e}^{i\phi/2}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathrm{e}^{-i\phi/2}\cos\left(\frac{\theta}{2}\right) \\ \mathrm{e}^{i\phi/2}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} = \frac{1}{2}|\vec{e_{\theta,\phi}}\rangle$$

Therefore the eigenstate is $|\vec{e_{\theta,\phi}}\rangle$ and the eigenvalue is $\frac{1}{2}$.

(3) The inefficient way is to use an arbitrary coordinate system.
The wise way is to say that without loss of generality we can take the $z$ direction along $\vec{e_1}$.

Let us call $\vec{e_2} = \vec{e}$ . In this coordinate system $\langle \vec{e_1} | \vec{e_2} \rangle$ is simply $\langle \uparrow | \vec{e} \rangle$.
If the angle of $\vec{e}$ with the $z$ direction is $\theta$, then we get immediately from part (1) that:

$$|\langle \uparrow | \vec{e} \rangle|^2 = \cos^2 \left( \frac{\theta}{2} \right)$$

Using the trigonometric identity

$$\cos^2 \left( \frac{\theta}{2} \right) = \frac{1}{2}(1 + \cos\theta)$$

We get that:

$$|\langle \vec{e_1} | \vec{e_2} \rangle|^2 = \frac{1}{2}(1 + \vec{e_1} \cdot \vec{e_2})$$