E409: The $SU(3)$ Group

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The problem:

Given here are the matrix representations of the eight $SU(3)$ generators :

$$t^1 = \frac{1}{2}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\ t^2 = \frac{1}{2}\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},\ t^3 = \frac{1}{2}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\ t^4 = \frac{1}{2}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$$

$$t^5 = \frac{1}{2}\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix},\ t^6 = \frac{1}{2}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\ t^7 = \frac{1}{2}\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\ t^8 = \frac{1}{2\cdot 3^{1/2}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$$

(1) Why are there 8 generators?
(2) Find the Structure Constant of the group.

(*) It might be helpful (recommanded) to use a mathematical software for the algebra.

The solution:

(1)

A theorm in group theory states that :
‘ *The rank of the algebra (number of generators) for the $SU_{(n)}$ groups is $n^2 - 1$* ’
We will prove this theorm for the general case, and demostrate it for $SU(3)$:

(A) Special Unitary group of dimension n (namly $SU(n)$) is a set of all $n \times n$ unitary matrices with determinant equals to one, $SU(n) = \{U | U^+ = U^{-1}, det(U) = 1\}$

(B) Since U is unitary it can be written as $U = e^{-iA}$, while A is an $n \times n$ Hermitian matrix.

(C) Since a determinant does not depend on the basis, we'll choose one in which A is diagonal (we show it for 3×3 matrices, but it of course generalizes to the $n \times n$ case):

$$A = \begin{pmatrix}a_1&0&0\\0&a_2&0\\0&0&a_3\end{pmatrix} \text{ Then } U \text{ will be: } U = \begin{pmatrix}e^{-ia_1}&0&0\\0&e^{-ia_2}&0\\0&0&e^{-ia_3}\end{pmatrix} \text{ and}$$
$$det(U) = 1 \Rightarrow e^{-ia_1+a_2+a_3} = 1 \Rightarrow a_1 + a_2 + a_3 = 0 \Rightarrow Tr[A] = 0$$

(D) A general $n \times n$ complex valued matrix has $2 \cdot n^2$ free parameters. The Hermitian property reduce this number to n^2 (Since $A = A^+ \Rightarrow A^T = A^*$. There are n real numbers along the diagonal and $(n^2 - n)/2$ complex numbers above the diagonal, the complex numbers below the diagonal are their conjugates). The zero trace condition lowers the number of free parameters in A farther to $p \equiv n^2 - 1$.

(E) In order to span a space of matrices which has p free parameters, p independent matrices should compound the base. This $\{G_i\}$ set is called the generator set and is compound of p generators. An infinity of possibilities for building a generator base exists, and any of them is an equal generator base of $SU(n)$.
For the $SU(3)$ case presented here there are $3^2 - 1 = 8$ free parameters, and therefore 8 generators.
The generator base presented here is the standard one.
It can be shown that these matrices are linearly independent of each other.

(2)
Structure Constants are defined in the relation $[t_i, t_j] = i \sum_k f^k_{i,j} t^k$ to be '$f_{i,j,k}$'. Please note the summation over k. This equation can describe the commutation relation as a linear addition (combination) of the original generators. This way the whole group's multiplication table can be calculated. In the SU(3) group $f_{i,j,k}$ is a rank three antisymmetric tensor, whose non-zero elements (and their antisymmetric permutation) are defined as:

$$\begin{bmatrix} ijk & f_{i,j,k} \\ 123 & 1 \\ 147 & 1/2 \\ 147 & 1/2 \\ 156 & -1/2 \\ (246 & 1/2 \\ 257 & 1/2 \\ 345 & 1/2 \\ 367 & -1/2 \\ 458 & \sqrt{3}/2 \\ 678 & \sqrt{3}/2 \end{bmatrix}$$

The derivation of this result appears at the enclosed *Mathematica* file

Extra information about $SU(3)$:

It is the set of all Unitary Matrices ($U^+ = U^{-1}$) with $Det = 1$. It can represent a 3 dimensional rotation in the complex plane. All unitary transformations (which can be implemented as an operation of a unitary matrix on a complex vector) preserve the length of a complex vector. Therefore it can be treated as an extension of an orthogonal transformations (=matrices...), which rotate a real-valued vector space and preserve its length. The unitray matrices with det=1 forms a group whose elements can be described using continous parameters. Therefore we name it a Lie group.

The generator concept is defined for Lie groups, and represent the effect of a diffrential change of the parameter(s). The number of generators is equivalent to the number of the basic parameters. For a certian group an infinity of generators sets, all with the same total number, can be defined.

The SU(3) group has been first implemented in physics in 1961 by Murray Gell-Mann and Yuval Nee'man to discribed the relationship between then-elemtary particles. Today this perspective has evolved to discribe the interactions of the strong force, between quarks.

A nice introduction for physicists about group theory can be found in:
Arfken George, Mathematical Methods For Physicists, 1985, Ch.IV

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