E407: Rotation group SU(2)

Submitted by: Gan Ofir and Alony Avraham

The problem:
The group $SU(2)$, has the same "multiplication table" as the group $SO(3)$.
The defining representation of that group includes all of the unitary marices 2×2 with determinent 1.

1. Explain why any of these marices can be written in the form $\exp(\Sigma_j \alpha_j S_j)$.
2. Choose the generators in the simplest way.
3. What are the structure constants of the group?
4. What will happen to the structure constants if we choose the generators differently?
5. Why is the group $U(2)$ is never mentioned in the booke or exercises?

The solution:
1. If we choose S_i to be any generator of the group, then by definition $e^{-i\epsilon S_i}$ is a matrix that represents an element in the group.
If S_1, S_2, S_3 generators of the group, then $e^{-i\epsilon a_1 S_1} e^{-i\epsilon a_2 S_2} e^{-i\epsilon a_3 S_3}$ is also represents an element in the group.
Based on the fact that ϵ is infinitesimal :

$$e^{-i\epsilon a_1 S_1} e^{-i\epsilon a_2 S_2} e^{-i\epsilon a_3 S_3} = (1 - i\epsilon a_1 S_1)(1 - i\epsilon a_2 S_2)(1 - i\epsilon a_3 S_3)$$

$$= 1 - i\epsilon a_1 S_1 - i\epsilon a_2 S_2 - i\epsilon a_3 S_3 + O(\epsilon^2)$$

$$= 1 - i\epsilon(a_1 S_1 + a_2 S_2 + a_3 S_3) = e^{-i\epsilon(a_1 S_1 + a_2 S_2 + a_3 S_3)}$$

Let us denote: $\alpha_i = -i\epsilon a_i$. and therefore we get $e^{\Sigma_j \alpha_j S_j}$ is a matrix that represents any element in the group.

2. The general formula for finding a rotation matrix

$$R = 1 - \sin \Phi M_n - (1 - \cos \Phi) M_n^2$$

By simplicity we choose matrices which obey the following condition:

$$S^n = \begin{cases} I & n \text{ even} \\ S & n \text{ odd} \end{cases}$$

It's easy to see that the Pauli matrices apply this condition, and therefore we choose

$$S_i = \frac{1}{2} \sigma_i$$

3. It follows

$$[\sigma_x, \sigma_y] = i\sigma_z \quad \text{etc.}$$

And generaly

$$[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k$$

4.First of all we will choose new generators which are a linear combination of the old generators:

$$G_1 = \sum_i \alpha_i S_i \quad G_2 = \sum_i \beta_i S_i \quad G_3 = \sum_i \gamma_i S_i$$

I order to keep the same structure contants we demand

$$[G_l, G_m] = G_l G_m - G_m G_l$$

$$= \sum_i \alpha_i S_i \sum_j \beta_j S_j - \sum_j \beta_j S_j \sum_i \alpha_i S_i$$

We remember that $S_i S_j - S_j S_i = i S_k$ thus:

$$\sum_{ijk} (\alpha_i \beta_j - \alpha_j \beta_i) i S_k = i \sum_k \gamma_k S_k$$

$$\Downarrow$$

$$\gamma_i = \epsilon_{ijk} \alpha_j \beta_k \quad \text{etc.}$$

The second demend is linear independent. we get three equations that can be written as follows:

$$\begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}$$

So we demand that :

$$\begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix} \neq 0$$

From those two demends we get the condition:

$$\gamma_2^2 \neq \gamma_1^2 + \gamma_3^2$$

5. $U(2)$ group includes $\boldsymbol{\sigma} + \hat{I}$.

Most of the physical operators in nature doesn't obey the multiplication table of $U(2)$ group, hence the $U(2)$ group has no use in physical applications.