Ex403: Dilitations

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The problem:

In analogy to translation and rotations, we define a dilitation using the transformation: $\mathbf{r}' = a\mathbf{r}$. We observe that the following relation holds: $\tau(a)\tau(b) = \tau(ab)$.

(1) Suggest a norm preserving realization over the function space for the dilitation operator.
(2) Whast does the dilitation operation look like in the momentum representation?
(3) Suggest a more convenient realization such that: $\tau(a)\tau(b)=\tau(a+b)$.
(4) Find the differential representation of the generator of dilitations G.
(5) Express G through p and r.
(6) Compute the commutators $[G,p],[G,r]$.
(7) Write down an explicit expression for $U^\dagger rU$ and $U^\dagger pU$ from the definition that you have found in 1 and 2, and deduce from this the commutation relations of 6.
(8) Show that the inverse is also correct, namely that the commutation relation $[G,r]$ implies the result of $U^\dagger rU$ and $U^\dagger pU$.

Remark on the solution by D.C.
It is best not to use Dirac notations in this exercise because the "measure" issue is tricky. One should remember that that Dirac basis $|x\rangle$ is not normalized is the standard way. The basis state

$$|x_0\rangle \quad \mapsto \quad \delta(x-x_0)$$

has an infinite norm and hence it is "living" outside of the Hilbert space. The normalization is not $\langle x_0|x_0\rangle = 1$ but rather

$$\langle x_1|x_2\rangle = \delta(x_1-x_2)$$

if one performs a dilation the result is

$$U|x_0\rangle \quad \mapsto \quad \frac{1}{\sqrt{a}}\delta((x/a)-x_0)$$

leading to

$$U|x_0\rangle = \sqrt{a}|ax_0>$$

where we used the well known identity $\delta(x/b) = b\delta(x)$. It is easy to get confused if you write e.g. $U|0\rangle = \sqrt{a}|0\rangle$. It look as if normalization is not preserved by this unitary transformation. This is not quite correct. The issue here is the "measure". If you like to avoid this mathematical discussion, simply avoid the Dirac notations in this context.

The solution:
We shall solve the problem for cartesian coordinates and then generalize the result.

(1) We define:

$$\tau(a)\psi(x) = A\psi(\frac{x}{a})$$

where we note that the 'moving trees' law has been applied. Assuming that $\psi(x)$ is normalized, then in order to preserve normalization we require:

$$A\int |\psi(\frac{x}{a})|^2 dx = 1$$

from which we obtain: $A = \frac{1}{\sqrt{a}}$, thus: $\tau(a)\psi(x) = \frac{1}{\sqrt{a}}\psi(\frac{x}{a})$.

(2) $\tau(a)\psi(k) = \tau(a)FT[\psi(x)] = FT[\tau(a)\psi(x)] = \frac{1}{\sqrt{a}}FT[\psi(\frac{x}{a})] = \sqrt{a}\psi(ak)$

(3)We define:

$$a = e^{\alpha}$$

From which we obtain:

$$\tau(a)\psi(x) = e^{-\frac{1}{2}\alpha}\psi(e^{-\alpha}x)$$

.

(4) We wish to find: $\langle x|G|\psi\rangle$. We calculate:

$$\langle x|U(\delta a)|\psi\rangle = \langle x|e^{-i\delta\alpha G}|\psi\rangle = \langle x|I - i\delta\alpha G|\psi\rangle = \frac{1}{\sqrt{e^{\alpha}}}\psi(e^{-\delta\alpha}x) =$$

$$(1 - \frac{1}{2}\delta\alpha)\psi((1 - \delta\alpha x)x) = (1 - \frac{1}{2}\delta\alpha)(\psi(x) - \delta\alpha x\frac{\partial\psi}{\partial x})$$

By regrouping and keeping only first order terms we get:

$$\psi(x) - \delta\alpha(x\frac{\partial\psi}{\partial x} + \frac{1}{2}\psi(x))$$

Comparing with the result of section 3 we obtain:

$$G \to -i(r\frac{\partial}{\partial x} + \frac{1}{2})$$

and generalizing to 3D we obtain:

$$G \to -i(\mathbf{r} \cdot \overline{\nabla} + \frac{1}{2})$$

(5) We know that $P \to -i\overline{\nabla}$, so: $G \to (\widehat{r} \cdot \widehat{p} - i\frac{1}{2}) = \frac{1}{2}(r \cdot p + p \cdot r)$.

(6) The obvious way is to use the result of section 5 and $[\widehat{r}, \widehat{p}] = i$:

$$[G, r] = -ir, [G, p] = -ip$$

We can verify the result for $[G, r]$ by calculating:

$$[\widehat{U}(\delta a), \widehat{r}]|r\rangle = \widehat{U}(\delta a)\widehat{r}|r\rangle - \widehat{r}\widehat{U}(\delta a)|r\rangle = r\widehat{U}(\delta a)|r\rangle - e^{-\delta\alpha}r\widehat{U}(\delta a)|r\rangle = r(1-e^{-\delta\alpha})(I - i\delta\alpha\widehat{G})|r\rangle = \delta\alpha r|r\rangle$$

where we keep only first order terms. Now: $[\widehat{U}(\delta a), \widehat{r}] = [I - i\delta\alpha\widehat{G}, \widehat{r}]$ so that finaly we get:

$$[\widehat{G}, \widehat{r}] = -i\widehat{r}$$

(7) Refering to section 1, we observe that:

$$U\psi(x) = \frac{1}{\sqrt{a}}\psi(\frac{x}{a}) = \frac{1}{\sqrt{a}}\langle\frac{x}{a}|\psi\rangle = \langle x|U|\psi\rangle \Rightarrow U^{\dagger}|x\rangle = \frac{1}{\sqrt{a}}|\frac{x}{a}\rangle$$

Therefore:

$$U|x\rangle \ = \sqrt{a}|ax\rangle$$

Now we can trivially obtain the desired result by applying:

$$U^\dagger x U|x\rangle = ax|x\rangle$$

Note that we can also obtain this result by applying directly on the wave function:

$$U^\dagger x U\psi(x) = U^\dagger x\frac{1}{\sqrt{a}}\psi(\frac{x}{a}) = ax\psi(x)$$

Using section 2, we follow the same procedure to obtain $U^\dagger pU = \frac{1}{a}p$

(8) To gain insight, let us calculate: $U^\dagger pU$ and $U^\dagger rU$. To this end we can calculate:

$$[\widehat{r}, \widehat{U}(\delta\alpha)] = -i\delta\alpha[\widehat{r}, \widehat{G}] = \delta\alpha r = \widehat{r}\widehat{U}(\delta\alpha) - \widehat{U}(\delta\alpha)\widehat{r}$$

Thus:

$$U(\delta\alpha)^\dagger rU(\delta\alpha) = \widehat{U}(\delta\alpha)^\dagger\delta\alpha r + r = r(1+\delta\alpha)$$

From which we conclude:

$$U^\dagger rU = re^\alpha$$

Similarly we obtain: $U^\dagger pU = pe^{-\alpha}$. We conclude that if we know the commutation relation of the generator of dilitations and a hermitian operator, we can calculate the dilitation transformation on the operator.