Ex3641: Electrons in finite potential well + Magnetic field

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The problem:

Consider a two-dimensional potential well LL with depth V_0, This means that inside the well $V(x,y) = V_0$. The walls of the well are very high and $V(x,y) = 0$ outside the well. Therefore, all electrons with energy $E > 0$ emerge out from the well. Assume that the remaining electrons in the well are spinless, have a mass M, and a charge e.

1. Write a condition for $L_{min} < L$ so that at least one electron could be located in the well.

   From now on assume $L >> L_{min}$.

2. Calculate the number of electrons N_0 that could be located in the well.

   Now a homogeneous magnetic field B is created perpendicular to the plane where the potential well is situated. In the next sections you can use known results for Landau levels (without any proof).

3. Find the maximum field $B = B_n$ in order to have n Landau levels full in the well. The number of particles in this section will be noted as N_n.

4. Find the ratio $\frac{N_n}{N_0}$.

5. Calculate the total energy of the system, E, with n Landau levels full.

6. Does the expression $-\frac{dE}{dB}$ has a meaning of magnetization? explain in one sentence.

Guidance:

- The boundary conditions of the well are of negligible influence on the spectrum, so it is possible to use known results derivated from Landau's solution.
- In section 5 the well's potential floor can be considered as a relative energy.

The solution:

(1)

$\mathcal{H} = \frac{1}{2M}(P_x^2 + P_y^2) + V(x) = \frac{1}{2M}(P_x^2 + P_y^2) - V_0$

The energy levels can be found as follows:

$E_{n_x,n_y} = \frac{1}{2M}[(\frac{\pi n_x}{L})^2 + (\frac{\pi n_y}{L})^2] - V_0 = \frac{1}{2M}(\frac{\pi}{L})^2(n_x^2 + n_y^2) - V_0$.

In order to have at least one electron in the well we have to look at the ground level, where $n_x = n_y = 1$, and demand $E_{1,1} < 0$ so the electron in that level would stay in the well:

$E_{1,1} = \frac{1}{M}(\frac{\pi}{L})^2 - V_0 < 0 \Longrightarrow L > \frac{\pi}{\sqrt{MV_0}}$

for L_{min} we demand equality, thus:

$L_{min} = \frac{\pi}{\sqrt{MV_0}}$

(2)

For $L >> L_{min}$ we know that there are electrons located inside the well.

Therefore, every state (n_x, n_y) has an energy $E < 0$ and must comply with:

$n_x^2 + n_y^2 < \frac{2ML^2V0}{\pi^2}$

This equation describes a disc in the n_x, n_y plane with it's center at $(n_x = 0, n_y = 0)$ and radius $R^2 = \frac{2ML^2V_0}{\pi^2}$, Remembering that $n_x, n_y > 0$ and $L >> L_{min}$ yields that N_0, representing the number of electrons in the well, is the area of a quarter circle with radius R as seen in Figure 1:

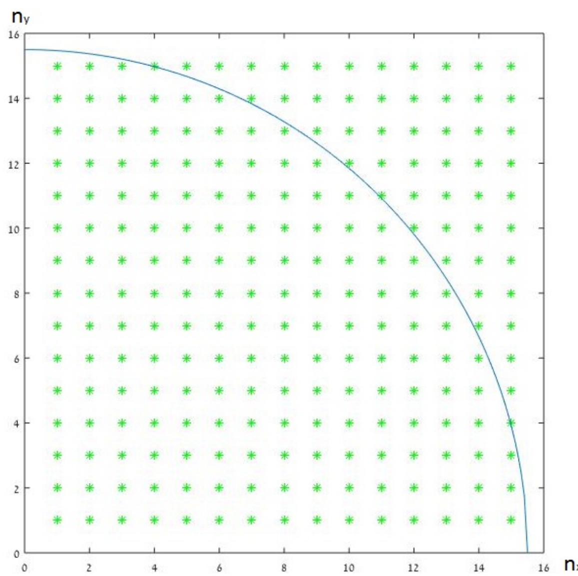

Figure 1: Number of occupied levels given a certain length

Thus:

$N_0 = \frac{\pi R^2}{4} = \frac{ML^2V_0}{2\pi}$

(3)

Adding the magnetic field changes the Hamiltonian, we will use Landau gauge for the vector potential $\vec{A} = (-By, 0, 0)$ in order to obtain $[\mathcal{H}, P_x] = 0$. So now the Hamiltonian will be:

$\mathcal{H} = \frac{1}{2M}(P_x + By)^2 + \frac{1}{2M}P_y^2 - V_0$

Since $[\mathcal{H}, P_x] = 0$ we can separate variables:

$\mathcal{H} = \frac{P_y^2}{2M} + \frac{1}{2M}(\frac{\pi n_x}{L} + By)^2 - V_0 = \frac{P_y^2}{2M} + \frac{B^2}{2M}(\frac{\pi n_x}{LB} + y)^2 - V_0$

This Hamiltonian is of a shifted harmonic oscillator in $\hat{y}$. Therefore its energy levels will be:

$E_{n_y} = \frac{B}{M}(n_y + \frac{1}{2}) - V_0$

In order for an electron to stay in the well we demand $E_{n_y} < 0$. A maximum magentic field with n Landau levels full is equivalent to have $n_y = n - 1$. therefore:

$E_{n_y=n-1} = \frac{B}{M}(n - \frac{1}{2}) - V_0) < 0$

$\Downarrow$

$B < \frac{MV_0}{n-\frac{1}{2}} \Rightarrow B_n = \frac{MV_o}{n-\frac{1}{2}}$

(4)

With n Landau levels full, each landau level has a degeneration of $g_{Landau} = \frac{L_x L_y B}{2\pi} = \frac{L^2 MV_o}{2\pi(n-\frac{1}{2})}$ states.

$N_n = n * g_{Landau} = \frac{nL^2 MV_o}{2\pi(n-\frac{1}{2})}$

$\Downarrow$

$\frac{N_n}{N_0} = \frac{\frac{nL^2 MV_o}{2\pi(n-\frac{1}{2})}}{\frac{ML^2 V_0}{2\pi}} = \frac{n}{n-\frac{1}{2}}$.

(5)

Using the number of states for every Landau level found in (4) and noting the energy of every occupied state as $E_i = \frac{B}{M}(i + \frac{1}{2})$. This makes for the total energy:

$E_{total} = \sum_{i=0}^{n-1} E_i * g_{Landau} = \sum_{i=0}^{n-1} \frac{B_n^2 L^2}{2\pi M}(i + \frac{1}{2}) = \frac{B_n^2 L^2 n^2}{4\pi M} = \frac{MV_0^2 L^2 n^2}{4\pi(n-\frac{1}{2})^2}$

(6)

the system is not isolated with a constant number of particles. A change in the magnetic field would cause a change in the number of electrons in the well which in turn affects energy of the system. Therefore, $-\frac{dE}{dB}$, does not represent magnetization.