E353: Hamiltonian of a particle in magnetic field of plane wave

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The problem:

Consider Particle withmass m with vector potential $\vec{A}(\vec{r}) = \vec{a}sin(\vec{k}\cdot\vec{r})$ where $\vec{a}$ and $\vec{k}$ perpendicular.

(1) Write the expression that corresponds to the Zeeman term in this case.
(2) Find what is the magnetic field $\vec{B}$. In particular find $\vec{B}(r=0)$.
(3) Assume that the region of interest is near $r=0$, so we have in the Hamiltonian a Zeeman term with $\widetilde{\vec{B}} = \vec{B}(r=0)$. Call it H_{int}^{Zeeman}.
(4) Do you have in the $r=0$ region $H_{int}^{Zeeman} = H_{int}$. Explain the difference.
Tip : Define $\widetilde{\vec{A}} = \frac{1}{2}(\widetilde{\vec{B}} \times \vec{r})$. is $\widetilde{\vec{A}}$ identical with $\vec{A}$ in the region $r=0$??? For simplicity you can take $\vec{k}$ in the $\hat{x}$ direction and $\vec{a}$ in the $\hat{y}$ direction.

The solution:

(1) firs we write general Hamiltonian :

$$H = \frac{1}{2m}(\vec{p}-\vec{A})^2 = \frac{1}{2m}(p^2 - (\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}) + A^2)$$

the Zeeman term

$$\frac{1}{2m}(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p})$$

substituting the given vector potential

$$\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p} = \vec{p}\cdot\vec{a}sin(\vec{k}\cdot\vec{r}) + \vec{a}sin(\vec{k}\cdot\vec{r})\cdot\vec{p} =$$

from the condition $\vec{a}\perp\vec{k}$ and that $\vec{a} = const.$

$$= 2\vec{p}\cdot\vec{a}sin(\vec{k}\cdot\vec{r})$$

So the expression corresponds to the Zeeman term is :

$$\frac{1}{m}\vec{p}\cdot\vec{a}sin(\vec{k}\cdot\vec{r})$$

(2) to find $\vec{B}$ we use the definition $\vec{B} = rot(\vec{A})$

$$\vec{B} = (\vec{k}\times\vec{a})cos(\vec{k}\cdot\vec{r})$$

now substituting $r=0$

$$\vec{B}(r=0) = (\vec{k}\times\vec{a})$$

(3) first we define

$$\widetilde{\vec{B}} = \vec{B}(r=0) = (\vec{k}\times\vec{a})$$

now in order to define the Zeeman term from the Hamiltonian H_{int}^{Zeeman} we first need to define $\widetilde{\vec{A}}$

$$\widetilde{\vec{A}} \equiv \frac{1}{2}(\widetilde{\vec{B}} \times \vec{r})$$

for simplicity from this moment we discuss only the specific choice $\vec{k} = k\hat{x}$; $\vec{a} = a\hat{y}$

$$\widetilde{\vec{B}} = ak\hat{z} \Rightarrow \widetilde{\vec{A}} = \frac{1}{2}ak(-y\hat{x} + x\hat{y})$$

because the form of $\widetilde{\vec{A}}$ we have $[\widetilde{\vec{A}}, \vec{p}] = 0$ so the Zeeman term become

$$H_{int}^{Zeeman} = \frac{1}{m}\vec{p} \cdot \widetilde{\vec{A}}$$

now substituting $\widetilde{\vec{A}}$ into the Zeeman term

$$H_{int}^{Zeeman} = \frac{1}{2m}ak(-p_x y + p_y x)$$

(4) lets take the Zeeman term from 1

$$H_{int} = \frac{1}{m}\vec{p} \cdot \vec{a}sin(\vec{k} \cdot \vec{r}) = \frac{1}{m}p_y a sin(kx)$$

now lets expend by Taylor near $r \approx 0$

$$H_{int}(r \approx 0) = \frac{1}{m}p_y a(kx - o(x^3)) = \frac{1}{m}akp_y x$$

so we have at $r \approx 0$:

$$H_{int}^{Zeeman} \neq H_{int}$$

the reason as in a moment will be shown is gauge choice , and in fact the Zeeman tems are equal in both cases.

because of the gauge freedom ($\widetilde{\widetilde{\vec{A}}} \Rightarrow \widetilde{\vec{A}} + \nabla\phi$) let us try to construct scalar function that will not effect $\vec{B}$ but make the $\vec{A} = \widetilde{\vec{A}}$

$$\phi = \frac{1}{2}akxy \Rightarrow \nabla\phi = \frac{1}{2}ak(y\hat{x} + x\hat{y})$$

now updating the vector potential

$$\widetilde{\widetilde{\vec{A}}} \equiv \frac{1}{2}(\widetilde{\vec{B}} \times \vec{r}) + \nabla\phi$$

$$\widetilde{\widetilde{\vec{A}}} = akx\hat{y}$$

now lets retry comparing

$$H_{int}^{Zeeman} = \frac{1}{m}akp_y x = H_{int}$$

so the conclusion is that the choise of vector potential must be consistent through the problem , in such a way that if we start working with one gauge , and decide to change for some reason the gauge , we must fix the new functions by some factors , or another gauge freedom , for example gradient of scalar function (but of course in such a way that the B field is not effected) in order not to change the phyisical conditions of the problem .