E315: Magnetic monopoles

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The problem:

Essume that there are magnetic monopoles with a charge e_m. Let's suppose that the magnetic field of the monopole is similar to the field on the tip of a very long solenoid on $z > 0$.

(1) What is the magnetic field of a single monopole?
(2) Show that $\vec{A} = \frac{e_m(1-\cos(\theta))}{r\sin(\theta)}\vec{e_\phi}$ is the suitable potential for $\vec{B}$ field.
(3) Does the potential well defined in the whole space?
(4) In what condition the presence of a solonoid will not be physicly felt?
(5) Define the potential $\vec{A}'$ which is well defined where $\vec{A}$ was not defined well.
(6) Where is the second potential not defined well?
(7) What is the gauge transformation?
(8) What is the connection between the wave function in the first gauge to the wave function in the second?
(9) What is the connection between the magnetic charge to the electrical charge that derive from the restriction on the wave function which is a single valued?

The solution:

(1) Magnetic field of a single monopole is:

$$\nabla(\vec{B}) = 4\pi\rho_m$$

Therefore:

$$\vec{B} = \frac{e_m}{r^2}\vec{e_r}$$

(2) We can see that:

$$B = \nabla \times \vec{A} = \frac{e_m}{r\sin(\theta)}\frac{\partial}{\partial\theta}\left(\frac{-\cos(\theta)}{r}\right)\vec{e_r} = \frac{e_m}{r^2}\vec{e_r}$$

(3) The pontetial is not defined well in the whole space. This pontential $\vec{A}$ is not defined well for $\theta \to \pi$.

(4) Because the magnetic field spreads like the magnetic field of an edge of solenoid, we can argue that we have a magnetic charge - e_m located at the tip of the solenoid. From the Gauss law we get $\Phi = 4\pi e_m$, therefore $e_m = \frac{\Phi}{4\pi}$. We dont want the influence of the flux inside the solenoid, so by using Aharonov-Bohm we demand that the flux should be: $\Phi = \frac{2\pi\hbar cn}{e}$, where n is integer. Therefore we get $e_m = \frac{\hbar cn}{2e}$ (which is the same solution we got in (9))

(5) Let's define the potential $\vec{A}$ which is well defined where $\vec{A}$ was not defined.
The problem point is: $\theta \to \pi$, so, our new potential $\vec{A}'$ will be defined well in this point:

$$\vec{A}' = -\frac{e_m(1+cos(\theta))}{rsin(\theta)}\vec{e_\phi}$$

(6) The second potential $\vec{A}'$ is not defined well for $\theta \to 0$.

(7) In the range $0 < \theta < \pi$ the both potentials are describes the same magnetic field so, we can write:

$$\vec{A}' = \vec{A} + \nabla\Lambda \to \vec{A}' - \vec{A} = \nabla\Lambda$$

$$\vec{A}' - \vec{A} = -\frac{2e_m}{r\sin(\theta)}\vec{e_\phi}$$

$$\nabla\Lambda = \frac{\partial\Lambda}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial\Lambda}{\partial\theta}\vec{e_\theta} + \frac{1}{r\sin(\theta)}\frac{\partial\Lambda}{\partial\phi}\vec{e_\phi} \rightarrow \Lambda = -2e_m\phi$$

And the gauge is: $\vec{A}' = \vec{A} + \nabla\Lambda$ when $\Lambda = -2e_m\phi$

(8) The wave function which compatibles to $\vec{A}$ is Ψ and the wave function which compatibles to $\vec{A}'$ is $\tilde{\Psi}$.
In order to satisfy the conditions: $H\Psi = ih\dot{\Psi}$ and $H'\tilde{\Psi} = ih\dot{\tilde{\Psi}}$ the connection between them has to be:

$$\tilde{\Psi} = \exp(-\frac{ie}{\hbar c}\Lambda)\Psi$$

In the section (6) we found that $\Lambda = -2e_m\phi$ then:

$$\tilde{\Psi} = \exp(\frac{2iee_m\phi}{\hbar c})\Psi$$

(9) Now we will find the connection between the magnetic charge to the electrical one. We know that wave function Ψ is a single valued, so in order $\tilde{\Psi}$ will be a single valued, $\exp\left(\frac{2iee_m}{\hbar c}\phi\right)$ has to be a single valued function too. From this reason: $\exp\left(\frac{2iee_m}{\hbar c}\phi\right) = \exp\left(\frac{2iee_m}{\hbar c}(\phi + 2\pi)\right)$.
Therefore:

$$\exp\left(\frac{2iee_m}{\hbar c}2\pi\right) = 1$$

And the answer is:

$$\frac{2ee_m}{\hbar c} = n \rightarrow e_m = \frac{\hbar c n}{2e}$$