Ex3121: Aharonov Bohm Ring

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The Problem:

a one-dimensional ring is connected by two one-dimensional wires, and has a magnetic flux of ϕ. The two wires separate the ring in such a way that it is composed of two arcs of lengths L_1, L_2:

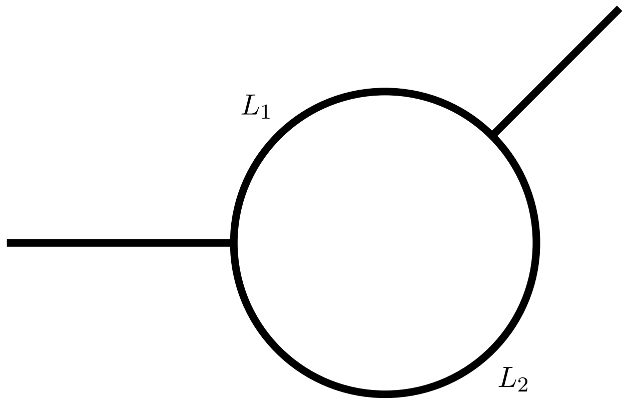

Figure 1: Aharonov-Bohm ring

Now, the question is, what would be the transmission coefficient for a wave coming in one terminal and exiting in the other terminal.

The Solution:

Inside the ring there's a magnetic flux therefore we would expect a vector potential A applied on the ring, and since we have a radial symmetry we would also expect A to be radial, therefore constant along the ring:

$$A = \frac{\phi}{L_1 + L_2} \tag{1}$$

Now that we know all about the potentials on the ring, we would like to solve shrodinger equation:

$$E\psi = \frac{1}{2m}\left[-i\frac{d}{dx} - A\right]^2 \psi$$

Where x is the arc distance along the wire from the left junction point. Now lets solve the equation using $\psi = e^{ik_{ring}x}$, and (1):

$$Ee^{ik_{ring}x} = \frac{1}{2m}\left[-i\frac{d}{dx} - \frac{\phi}{L_1 + L_2}\right]^2 e^{ik_{ring}x}$$

And we find that:

$$k_{UpperArc} = \pm\sqrt{2mE} - \frac{\phi}{L_1 + L_2} = \pm k - \frac{\phi}{L_1 + L_2}$$

$$k_{LowerArc} = \pm\sqrt{2mE} + \frac{\phi}{L_1 + L_2} = \pm k + \frac{\phi}{L_1 + L_2}$$

The difference in the solutions for the two arcs is the result of picking the left junction as our origin, while x grows positively along each arc, and A, the vector potential, is either along the x axis, or against it.

We will now give names for the different wave functions:

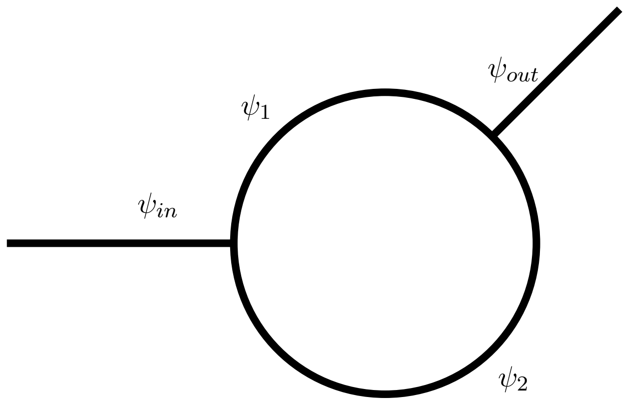

Figure 2: Names of the wave functions

So we have the following equations:

$$\psi_{in} = A_{in} exp\left(-ikx\right) + B_{in} exp\left(ikx\right)$$

$$\psi_1 = A_1 exp\left(i\left[-k - \frac{\phi}{L_1 + L_2}\right]x\right) + B_1 exp\left(i\left[+k - \frac{\phi}{L_1 + L_2}\right]x\right)$$

$$\psi_2 = A_2 exp\left(i\left[-k + \frac{\phi}{L_1 + L_2}\right]x\right) + B_2 exp\left(i\left[+k + \frac{\phi}{L_1 + L_2}\right]x\right)$$

$$\psi_{out} = A_{out} exp\left(-ikx\right) + B_{out} exp\left(ikx\right)$$

Now, let us recall that the transmittance is defined as:

$$T = \left|\frac{A_{out}}{B_{in}}\right|^2$$

To find the transmittance we first wish to express the values of the wave function on the two junctions. So, for $x = 0$ we have(remember, x is the distance from the origin of the branch):

$$\psi_{in}(0) = A_{in} + B_{in}$$

$$\psi_1(0) = A_1 + B_1$$

$$\psi_2(0) = A_2 + B_2$$

$$\psi_{out}(0) = A_{out} + B_{out}$$

and for $x = L_1, L_2$ respectively, which is only relevant for the two arcs, we have:

$$\psi_1(L_1) = A_1 \cdot exp\left(iL_1\left[-k - \frac{\phi}{L_1 + L_2}\right]\right) + B_1 \cdot exp\left(iL_1\left[k - \frac{\phi}{L_1 + L_2}\right]\right)$$

$$\psi_2(L_2) = A_2 \cdot exp\left(iL_2\left[-k + \frac{\phi}{L_1 + L_2}\right]\right) + B_2 \cdot exp\left(iL_2\left[k + \frac{\phi}{L_1 + L_2}\right]\right)$$

To find the transmittance we will use the 3-branch S matrix:

$$\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

So that we have the relation between the wave functions coming into the junction and the waves coming out of the junction:

$$S \cdot \begin{pmatrix} A_{in} \\ A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} B_{in} \\ B_1 \\ B_2 \end{pmatrix}, S \cdot \begin{pmatrix} A_{out} \\ B_1 \cdot exp\left(iL_1\left[k - \frac{\phi}{L_1+L_2}\right]\right) \\ B_2 \cdot exp\left(iL_2\left[k + \frac{\phi}{L_1+L_2}\right]\right) \end{pmatrix} = \begin{pmatrix} B_{out} \\ A_1 \cdot exp\left(iL_1\left[-k - \frac{\phi}{L_1+L_2}\right]\right) \\ A_2 \cdot exp\left(iL_2\left[-k + \frac{\phi}{L_1+L_2}\right]\right) \end{pmatrix}$$

Although we expressed the coefficient B_{out} in the equation above, we know that there can be no reflection in this part of the system, so we deduce that $B_{out} = 0$. Also, for convenience we will fix $B_{in} = 1$, since T will not change for different amplitudes of the input wave, so we have that:

$$S \cdot \begin{pmatrix} A_{in} \\ A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 1 \\ B_1 \\ B_2 \end{pmatrix}, S \cdot \begin{pmatrix} A_{out} \\ B_1 \cdot exp\left(iL_1\left[k - \frac{\phi}{L_1+L_2}\right]\right) \\ B_2 \cdot exp\left(iL_2\left[k + \frac{\phi}{L_1+L_2}\right]\right) \end{pmatrix} = \begin{pmatrix} 0 \\ A_1 \cdot exp\left(iL_1\left[-k - \frac{\phi}{L_1+L_2}\right]\right) \\ A_2 \cdot exp\left(iL_2\left[-k + \frac{\phi}{L_1+L_2}\right]\right) \end{pmatrix}$$

This is of course a system of 6 equations with 6 variables, therefore solvable. here is another representation of the system:

$$\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} A_{in} \\ A_1 \\ A_2 \\ A_{out} \\ B_1 \cdot exp\left(iL_1\left[k - \frac{\phi}{L_1+L_2}\right]\right) \\ B_2 \cdot exp\left(iL_2\left[k + \frac{\phi}{L_1+L_2}\right]\right) \end{pmatrix} = \begin{pmatrix} 1 \\ B_1 \\ B_2 \\ 0 \\ A_1 \cdot exp\left(iL_1\left[-k - \frac{\phi}{L_1+L_2}\right]\right) \\ A_2 \cdot exp\left(iL_2\left[-k + \frac{\phi}{L_1+L_2}\right]\right) \end{pmatrix}$$

Solving the system, we can find A_{out}:

$$A_{out} = -\frac{4e^{\frac{iL_2\phi}{L_1+L_2}}\left(\left(-1 + e^{2ikL_1}\right)e^{i(kL_2+\phi)} - e^{ikL_1} + e^{ik(L_1+2L_2)}\right)}{4\left(1 + e^{2i\phi}\right)e^{ik(L_1+L_2)} - 9e^{i(2k(L_1+L_2)+\phi)} + e^{i\phi}\left(e^{2ikL_1} + e^{2ikL_2} - 1\right)}$$

The transmittance is defined as:

$$T = \left|\frac{A_{out}}{B_{in}}\right|^2 = |A_{out}|^2$$

Therefore, the transmittance is:

$$T = -\frac{16(-4\cos(\phi)\sin(kL_1)\sin(kL_2)+\cos(2kL_1)+\cos(2kL_2)-2)}{16\cos(\phi)(\cos(k(L_1-L_2))-5\cos(k(L_1+L_2)))-10\cos(2kL_1)+\cos(2k(L_1-L_2))-10\cos(2kL_2)+9\cos(2k(L_1+L_2))+16\cos(2\phi)+58}$$

After switching to the variables $L = L_1 + L_2, d = L_1 - L_2$ we get:

$$T = \frac{1 + \cos(\phi)\cos(dk) - \cos(kL)[\cos(dk) + \cos(\phi)]}{1 + \frac{1}{16}[\cos(dk) + 4\cos(\phi) - \cos(kL)][\cos(dk) + 4\cos(\phi) - 9\cos(kL)]}$$