E313: Aharonov-Bohm Interferometer

Submitted by: Alisa Bondarev 316971738, Avner Segal 036769412, Arthur Shulkin 310159025

The problem:

A charged particle (charge e, mass m) is sent from the left into an Aharonov-Bohm Interferometer (as in problem 312), inside the ring there is a flux Φ, it is known that for each vertix we have "Kirchhoff's law" $\Sigma\partial\psi = \lambda\psi$ where λ is given, and also the particle's wave-number k and the length of each half-ring L are given. Calculate the transmission coefficient T(Φ, λ,k,L) for the interferometer.
(Hint: There is no nead to calculate a scattering matrix S for the vertix)

The solution:

We will define each vertix as the orgin of three axes with the positive directions defind as outwards to all three of them. We notice that in the lines which connect the vertices there are two sets of axes meeting and have opposite positive directions, we will use extra two functions to help us compute the wave-function at every point. We define:

$$\psi_I = Ae^{+ikx} + Be^{-ikx}$$

$$\psi_{II} = Ce^{-ikx}$$

$$\psi_{III} = De^{i(k+\frac{\Phi}{2L})x} + Ee^{i(-k+\frac{\Phi}{2L})x}$$

$$\psi_V = Fe^{i(k-\frac{\Phi}{2L})x} + Ge^{i(-k-\frac{\Phi}{2L})x}$$

The two "extra" functions we define are:

$$\psi_{IV}(x) = \psi_{III}(L-x)$$

$$\psi_{VI}(x) = \psi_V(L-x)$$

This functions are the same as the two equivalent ψ_{III}, ψ_V, but they are oriented in the positive direction in the sense of the axis going out from the right vertix.
From the continuity condition in the frist vertix we get:

$$\psi_I(0) = A + B = \psi_1$$

$$\psi_{III}(0) = D + E = \psi_1$$

$$\psi_V(0) = F + G = \psi_1$$

From the continuity condition in the second vertix:

$$\psi_{II}(0)C = \psi_2$$

$$\psi_{IV}(0) = De^{i(k+\frac{\Phi}{2L})x} + Ee^{i(-k+\frac{\Phi}{2L})x} = \psi_2 \Rightarrow$$

$$De^{ikL} + Ee^{-ikL} = \psi_2 e^{-i\frac{\Phi}{2}}$$

$$\psi_{VI}(0) = Fe^{i(k-\frac{\Phi}{2L})x} + Ge^{i(-k-\frac{\Phi}{2L})x} = \psi_2 \Rightarrow$$

$$Fe^{ikL} + G\exp^{-ikL} = \psi_2 e^{i\frac{\Phi}{2}}$$

The condition $\Sigma\partial\psi = \lambda\psi$ ("Kirchhoff's Law") on the two vertices gives:

$$ikA - ikB + i(k + \frac{\Phi}{2L})D + i(-k + \frac{\Phi}{2L})E + i(k - \frac{\Phi}{2L})F + i(-k - \frac{\Phi}{2L})G = \lambda\psi_1$$

$$-ikC - i(k + \frac{\Phi}{2L})De^{i(k+\frac{\Phi}{2L})L} - i(-k + \frac{\Phi}{2L})Ee^{i(-k+\frac{\Phi}{2L})L} - i(k - \frac{\Phi}{2L})Fe^{i(k-\frac{\Phi}{2L})L} - i(-k - \frac{\Phi}{2L})Ge^{i(-k-\frac{\Phi}{2L})L} = \lambda\psi_2$$

By simple algebra we have:

$$k(A - B) + k(D - E) + k(F - G) = -i\lambda\psi_1$$

$$-kC - ke^{i\frac{\Phi}{2}}(De^{ikl} - Ee^{-ikl}) - ke^{-i\frac{\Phi}{2}}(Fe^{ikl} - Ge^{-ikl}) = -i\lambda\psi_2$$

From the equations of continuity (in both vertices) we have:

$$A = \psi_1 - B$$

$$D = \psi_1 - E$$

$$F = \psi_1 - G$$

$$C = \psi_2$$

$$De^{ikl} + Ee^{-ikl} = \psi_2 e^{-i\frac{\Phi}{2}}$$

$$Fe^{ikl} + Ge^{-ikl} = \psi_2 e^{i\frac{\Phi}{2}}$$

Setting this into the equations of "Kirchhoff's Law" we have:

$$2kA + 2kD + 2kF + (3k + i\lambda)\psi_1 = 0$$

$$-2ke^{i(kL+\frac{\Phi}{2})}D - 2ke^{i(kL-\frac{\Phi}{2})}F - (3k - i\lambda)\psi_2 = 0$$

From these equations we can also get:

$$D = \frac{\psi_1 - \psi_2 e^{i(kL-\frac{\Phi}{2})}}{(1 - e^{i2kL})}$$

$$F = \frac{\psi_1 - \psi_2 e^{i(kL+\frac{\Phi}{2})}}{(1 - e^{i2kL})}$$

And thus have:

$$A2k + \frac{\psi_1 - \psi_2 e^{i(kL-\frac{\Phi}{2})}}{1 - e^{2ikL}}2k + \frac{\psi_1 - \psi_2 e^{i(kL+\frac{\Phi}{2})}}{1 - e^{2ikL}}2k + (3k - \frac{\lambda}{i})\psi_1 = 0$$

$$-\frac{\psi_1 - \psi_2 e^{i(kL-\frac{\Phi}{2})}}{1 - e^{2ikL}}2ke^{i(kL+\frac{\Phi}{2})} - \frac{\psi_1 - \psi_2 e^{i(kL+\frac{\Phi}{2})}}{1 - e^{2ikL}}2ke^{i(kL-\frac{\Phi}{2})} - (3k + \frac{\lambda}{i})\psi_2 = 0$$

Using the known fact that $\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ we have:

$$A + i\frac{2\psi_1 - \psi_2 e^{i(kL-\frac{\Phi}{2})} - \psi_2 e^{i(kL+\frac{\Phi}{2})}}{2\sin(kL)}e^{-ikL} + (\frac{3}{2} + i\frac{\lambda}{k})\psi_1 = 0$$

$$i\frac{\psi_1 - \psi_2 e^{i(kL-\frac{\Phi}{2})}}{2\sin(kL)}e^{i\frac{\Phi}{2}} + i\frac{\psi_1 - \psi_2 e^{i(kL+\frac{\Phi}{2})}}{2\sin(kL)}e^{-i\frac{\Phi}{2}} + (\frac{3}{2} - i\frac{\lambda}{k})\psi_2 = 0$$

Now using also that $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$ we get:

$$A + i\frac{\psi_1 e^{-ikL} - \psi_2\cos(\frac{\Phi}{2})}{\sin(kL)} + (\frac{3}{2} + i\frac{\lambda}{k})\psi_1 = 0$$

$$i\frac{\psi_1\cos(\frac{\Phi}{2}) - \psi_2 e^{ikL}}{\sin(kL)} + (\frac{3}{2} - i\frac{\lambda}{k})\psi_2 = 0$$

The second equation yields:

$$\psi_1 = \frac{e^{ikL} + (\frac{3}{2}i + \frac{\lambda}{k})\sin(kL)}{\cos(\frac{\Phi}{2})}\psi_2$$

And setting it into the first one we have (Remembering that $\psi_2 = C$):

$$\frac{A}{C} = -i\frac{(\frac{e^{ikL}+(\frac{3}{2}i+\frac{\lambda}{k})\sin(kL)}{\cos(\frac{\Phi}{2})})e^{-ikL} - \cos(\frac{\Phi}{2})}{\sin(kL)} - (\frac{3}{2}i + \frac{\lambda}{k})\frac{e^{ikL} + (\frac{3}{2}i + \frac{\lambda}{k})\sin(kL)}{\cos(\frac{\Phi}{2})}$$

Seperating to Real and Imaginery parts gives:

$$Re(\frac{A}{C}) = \frac{(\frac{3}{4}\sin(2kL) - \frac{\lambda}{k}\sin^2(kL) + \frac{3}{2}\cos(kL) + (\frac{9}{4} - \frac{\lambda}{k} - \frac{\lambda^2}{k^2})\sin(kL))}{\sin(kL)\cos(\frac{\Phi}{2})}$$

$$Im(\frac{A}{C}) = \frac{(-\frac{\lambda}{2k}\sin(2kL) - \frac{3}{2}\sin^2(kL) + \frac{\lambda}{k}\cos(kL) + \frac{6\lambda+3k}{2k}\sin(kL) - \sin^2(\frac{\Phi}{2}))}{\sin(kL)\cos(\frac{\Phi}{2})}$$

And thus the Transmission coefficient is:

$$T(\Phi, \lambda, k, L) = |\frac{A}{C}|^2 = (Re(\frac{A}{C})^2 + (Im(\frac{A}{C}))^2$$

We compute it with A and not B because the positive directions from the righthand vertix is to the right and the incoming particle is the e^{ikx} one.