E309: Particle Beatwin Two Concentrical Cylinders

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The problem:

A charged particle placed between two concentric cylinder with length l and radius r_a and r_b

(1) Find and expression for the energy level of the particle.
(2) Let assume that through the small cylinder there is a magnetic flux, show that even if the particle moves where the magnetic field is zero, his energy levels has changed.

The solution:

(1) Lets write the Hamiltonian:

$$\widehat{\mathcal{H}} = \frac{\widehat{p}^2}{2M}$$

The Laplasian in the cylindrical coordinates is:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

We will use the constants of motion $\widehat{p}_z$ and L_z, so in the cylindrical coordinates we can write the Hamiltonian like:

$$\mathcal{H} = \frac{1}{2M}\left(p_r^2 + \frac{1}{r^2}L_z^2 + p_z^2\right)$$

We know that:

$$[\mathcal{H}, L_z] = 0 \qquad \Rightarrow \qquad L_z = m$$

$$[\mathcal{H}, |p_z|] = 0 \qquad \Rightarrow \qquad |p_z| = \frac{\pi}{l}n_z$$

Now we can write the block Hamiltonian:

$$\mathcal{H}^{(m,n_z)} = \frac{1}{2M}\left(p_r^2 + \frac{1}{r^2}m^2 + \left(\frac{\pi}{l}n_z\right)^2\right)$$

Then the radial part we will solve whith Bessel equation:

$$R(r) = AJ_m\left(kr\right) + BN_m\left(kr\right)$$

The solution of the Bessel equation is ν (the radial quantum number). Then the energy is:

$$E_{m,n_z,\nu} = \frac{\hbar^2}{2M}\left(k_{m,\nu}^2 + \left(\frac{\pi}{l}n_z\right)^2\right)$$

(2) For the condition:

$$\vec{A}=\frac{\Phi}{2\pi r}\widehat{\varphi} \quad , \qquad r>r_a$$

The Hamiltonian will be:

$$\widehat{\mathcal{H}}=\frac{1}{2M}\left(\widehat{p}-e\vec{A}\right)^2$$

The gradient in the cylindrical coordinates is:

$$\vec{\nabla}=\frac{\partial}{\partial r}\widehat{r}+\frac{1}{r}\frac{\partial}{\partial\varphi}\widehat{\varphi}+\frac{\partial}{\partial z}\widehat{z}$$

$$\left(\widehat{p}-e\vec{A}\right)^2=p_r^2+\frac{1}{r^2}\left(L_z-\frac{e\Phi}{2\pi}\right)^2+p_z^2$$

We will define constants of motion $\widehat{p}_z$ and $m_{eff}=\left(L_z-\frac{e\Phi}{2\pi}\right)$, lets sign m_{eff} like the m:

$$\mathcal{H}^{(m,n_z)}=\frac{1}{2M}\left(p_r^2+\frac{1}{r^2}m^2+\left(\frac{\pi}{l}n_z\right)^2\right)$$

From the solution of the Bessel equation we got:

$$R(r)=AJ_m\left(kr\right)+BN_m\left(kr\right)$$

R(r) = A J$_m(kr)+BN_m(kr)Thentheenergyis:$

$$E_{m,n_z,\nu}=\frac{\hbar^2}{2M}\left(k_{m,\nu}^2+\left(\frac{\pi}{l}n_z\right)^2\right)$$