E304: Particle in one dimensional ring + scatter

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The problem:

a particle is placed with mass m and electric charge e in one dimensional ring with length L, the flux through the ring is Φ .
in addition there is in the ring a scattering potential $V(x) = u\delta(x)$.

(1) Find a close equation for the eigenenergys $E_n(\Phi)$ (no need to solve it).
(2) Show that in the limit $u \to \infty$ we will obtain the spectrum of an infinite potential well.
(3) Draw a diagram of the energies leves vs. the flux, in the limit that $u \to 0$.
(4) Explain (qualitative) what is the primary effect of the scatter (when present) in the diagram.

The solution:

(1) The Hamiltonian is

$$H = \frac{\left(p - \frac{e\Phi}{L}\right)^2}{2m} + u\delta(x)$$

for $0 < x < L$ we saw in the calss that the general solusion is $\psi = Ae^{ik_+x} + Be^{ik_-x}$

when $k_\pm = \pm k + e\frac{\Phi}{L}$

and $k = \sqrt{2mE}$

the Boundary conditions are :
1. without the scatter potential: $\psi(0) = \psi(L)$
2. with the presence of the scatter potential: $\psi'(0) - \psi'(L) = 2mu\psi(0)$
,we take $\hbar = 1$ for convenient.

from 1 we obtain : $A + B = Ae^{ik_+L} + Be^{ik_-L} \Rightarrow A = -B\frac{1-e^{ik_-L}}{1-e^{ik_+L}}$

from 2 we obtain : $(ik_+A + ik_-B) - \left(ik_+Ae^{ik_+L} + ik_-Be^{ik_-L}\right) = 2mu(A+B)$

so now we have this too equations :

$$\Rightarrow \begin{cases} A = -B\frac{1-e^{ik_-L}}{1-e^{ik_+L}} \\ ik_+A\left(1-e^{ik_+L}\right) + ik_-B\left(1-e^{ik_-L}\right) = 2mu(A+B) \end{cases}$$

by placeing 1 in to 2, we obtain :

$$ik_+\left(-B\left(1-e^{ik_-L}\right)\right) + ik_-\left(-B\left(1-e^{ik_-L}\right)\right) = 2mu\left(-B\frac{1-e^{ik_-L}}{1-e^{ik_+L}} + B\right)$$

after a little algebra we will obtain :

$$i(k_- - k_+)\left(-2i\sin\left(k_-\frac{L}{2}\right)\right)\left(-2i\sin\left(k_+\frac{L}{2}\right)\right) = 2mu\left(e^{i(k_--k_+)\frac{L}{2}} - e^{-i(k_--k_+)\frac{L}{2}}\right)$$

$$\Rightarrow (k_- - k_+)\sin\left(\frac{L}{2}k_-\right)\sin\left(\frac{L}{2}k_+\right) = mu\sin\left(\frac{L}{2}(k_+ - k_-)\right)$$

finally we will place the expression for $k_{\pm}$ in this equation, and we will obtain:

$\sin\left(\frac{kL}{2}-\frac{e\Phi}{2}\right)\sin\left(\frac{kL}{2}+\frac{e\Phi}{2}\right)=\frac{mu}{2k}\sin\left(kL\right)$

by using the product of trigonometric function we will get:

$\cos\left(e\Phi\right)=\frac{mu}{k}\sin\left(kL\right)+\cos\left(kL\right)$

(2)

in the limit $u\rightarrow\infty$, we see that by division both sides of the equation by u , the flux become insignificant.

$\sin\left(kL\right)=0$

$kL=\pi n$

$k=\sqrt{2mE}$

$E=\frac{1}{2m}k^{2}=\frac{1}{2m}\left(\frac{\pi n}{L}\right)^{2}$

this energy levels are those of an infinite potential well.

(3)

in the limit $u\rightarrow 0$, $\Phi\neq 0$,we will obtain:

$\cos\left(e\Phi\right)-\cos\left(kL\right)=0$

$\sin\left(\frac{kL+e\Phi}{2}\right)\sin\left(\frac{kL-e\Phi}{2}\right)=0$

$k_{anticlockwise}\left(n\right)=\frac{2\pi n}{L}-\frac{e\Phi}{L}$ when n is: $n=0,1,2,3....$

$k_{clockwise}\left(n\right)=\frac{2\pi n}{L}+\frac{e\Phi}{L}$ when n is: $n=0,1,2,3....$

both of the k fulfil the relation : $k=\sqrt{2mE}\Rightarrow E=\frac{1}{2m}k^{2}$

$E_{anticlockwise}\left(n\right)=\frac{1}{2m}\left(\frac{2\pi n}{L}-\frac{e\Phi}{L}\right)^{2}$ when n is: $n=0,1,2,3....$

$E_{clockwise}\left(n\right)=\frac{1}{2m}\left(\frac{2\pi n}{L}+\frac{e\Phi}{L}\right)^{2}$ when n is: $n=0,1,2,3....$

we see that we get two sets of energys, this happen because of the flux, meaning that the flux remove the degeneration. for a ring with no flux, the two k states: $|k_{anticlockwise}\rangle\mapsto e^{ikx}$ and $|k_{clockwise}\rangle\mapsto e^{-ikx}$ are the transparent of one another and ther for they are degenerat, giving the same energys value. the flux brok the symmetry, and remove the degeneration, this is why we now have tow sets of energys. the over-all energy tsates are : $E_{n}=\frac{1}{2m}\left(\frac{2\pi n}{L}-\frac{e\Phi}{L}\right)^{2}$ when n is : $n=0,\pm1,\pm2,\pm3...$ this is the same result as seen in class.

(4)

The presence of the scatter cause a splitting in the energy levels.

Addition by D.C.

First remark about the solution of (3). The equation is

$$\cos(k_E L) = \cos(e\Phi)$$

leading to

$$k_E L = \pm e\Phi + 2\pi n$$

By definition $k_E = (2mE)^{1/2}$ is non-negative so n is restricted accordingly. But because we square k_E to get E we can write the final result as one expression with unrestricted range for n.

$$E_n = \frac{1}{2mL^2}(2\pi n - e\Phi)^2$$

For those who know what is the S matrix of a delta function from Ex255 there is a much simpler way to get the eigenvalue equation. In the scattering approach the ring is opened at some arbitrary point and the S matrix of the open segment is specified. It is more convenient to use the row-swapped matrix, such that the transmission amplitudes are along the diagonal:

$$\tilde{\mathbf{S}} = \mathrm{e}^{i\gamma}\begin{pmatrix} \sqrt{g}\mathrm{e}^{i\phi} & -i\sqrt{1-g}\mathrm{e}^{-i\alpha} \\ -i\sqrt{1-g}\mathrm{e}^{i\alpha} & \sqrt{g}\mathrm{e}^{-i\phi} \end{pmatrix} \tag{1}$$

The periodic boundary conditions imply the equation

$$\begin{pmatrix} A \\ B \end{pmatrix} = \tilde{\mathbf{S}}\begin{pmatrix} A \\ B \end{pmatrix} \tag{2}$$

which has no-trivial solution if and only if

$$\det(\tilde{\mathbf{S}}(E) - \mathbf{1}) = 0 \tag{3}$$

Using

$$\det(\tilde{\mathbf{S}} - I) = \det(\tilde{\mathbf{S}}) - \mathrm{trace}(\tilde{\mathbf{S}}) + 1 \tag{4}$$

$$\det(\tilde{\mathbf{S}}) = (\mathrm{e}^{i\gamma})^2 \tag{5}$$

$$\mathrm{trace}(\tilde{\mathbf{S}}) = 2\sqrt{g}\mathrm{e}^{i\gamma}\cos\phi \tag{6}$$

we get

$$\cos(\gamma(E)) = \sqrt{g(E)}\cos(\phi) \tag{7}$$

where

$$g(E) = \left[1 + \left(\frac{\mathsf{m}}{\hbar^2 k_E}u\right)^2\right]^{-1} \tag{8}$$

and

$$\gamma(E) = k_E L - \arctan\left(\frac{\mathsf{m}}{\hbar^2 k_E}u\right) \tag{9}$$

Note that the free propagation in the lead is included in γ. In order to find the eigen-energies we plot both sides as a function of E. The left hand side oscillates between -1 and $+1$, while the right hand side may have a smaller amplitude.