Ex3020: Particle in 1D ring

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The problem:

Consider a particle with mass m and charge e, on a 1D ring with length L. The flux through the ring is Φ.

(1) Writh the vector potential $\overrightarrow{A}$ considering homogeneous gauge (”old” gauge).
(2) Writh the vector potential $\overrightarrow{A}$ considering delta gauge (”new” gauge).
(3) In the case of delta gauge, explain how the matching condition can be writen in S-matrix formalism.
(4) Find the eigenstates $|n(\Phi)\rangle$ in the old gauge and in the new gauge.
(5) What is the gauge transformation to the new gauge states?
Make sure the eigenenergies $En(\Phi)$ are the same for both gauges.

The solution:

(1) By difinition: $\oint \overrightarrow{A} \cdot d\overrightarrow{l} = \Phi$

In old gauge, A is homogeneous all over the ring, which yields: A=$\frac{\Phi}{L}$.

(2) In new gauge, A is localized at one point, which yields: $A(x) = \Phi \cdot \delta(x)$.

(3) When moves clockwise, the particle gains geometric phase of Φ at x=0.
The matching conditions for a ring is:

$\Psi(0) = e^{ie\Phi}\Psi(L) \Rightarrow$A=A$e^{i(kL+e\Phi)}$

When moves counterclockwise, it gains eometric phase of $-e\Phi$ at x=0. now it useful to take the condition on the derivations:

$\Psi'(L) = e^{-ie\Phi}\Psi'(0) \Rightarrow -ikBe^{(-ikL)} = -ikBe^{-ie\Phi}$ or $B = Be^{i(kL-e\Phi)}$.

Now we can write these results in S-matrix formalism:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 & e^{i(kL+e\Phi)} \\ e^{i(kL-e\Phi)} & 0 \end{pmatrix} \begin{pmatrix} B \\ A \end{pmatrix}$$

(4) In the old guege, the Hemiltonian is: $\mathcal{H} = \frac{1}{2m}\left(\hat{p} - \frac{e\Phi}{L}\right)^2$.

In this Hemiltonian, p is a constant of motion, so the eigenstates of $\mathcal{H}$ are the momentum states $|k_n\rangle$ where:

$k_n = \frac{2\pi}{L}n$ n=0,$\pm1, \pm2, ...$

And the eigenenergies:

$E_n = \frac{1}{2m}\left(\frac{2\pi}{L}n - \frac{e\Phi}{L}\right)^2$

In the new guege, however, the solution for the equations we wrote in (3) is:

$k_{\pm n}L \pm e\Phi = 2\pi$ n n=0,1,2,....

$k_{\pm n} = \frac{2\pi}{L}n \mp \frac{e\Phi}{L}$

And because $k_n = \sqrt{(2mE)}$ we will get:

$E_n = \frac{1}{2m}\left(\frac{2\pi}{L}n - \frac{e\Phi}{L}\right)^2$ n=0,$\pm1, \pm2, ...$

As we got also in the old gauge.

(5)Let us define a new basis:

$|\widetilde{x}\rangle = e^{-i\Lambda(x)}|x\rangle$

Where:

$$\Lambda = \begin{cases} \frac{-e\Phi}{L}x & x \neq 0 \\ \frac{-e\Phi}{L}x + \Phi x & x = 0 \end{cases}$$

and because:

$\widetilde{A}(x) = A(x) + \frac{d}{dx}\Lambda(x)$

we will get:

$$\widetilde{A}(x) = \begin{cases} 0 & x \neq 0 \\ \Phi & \text{x=0} \end{cases}$$