# E0292: Cross section and optical theorem in 2D using the T matrix formalism

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**The problem:**

In this problem you are requested to derive the scattering state and the optical theorem in 2D

(1) Write down the free wave function: $|E,\varphi\rangle$. keep it normalized.
(2) Write down the Green function.
(3) Show that the asymptotic behavior of the wave function can be written as: $\Psi(r) = \Phi^{k_0}(r) + if(\varphi)\frac{e^{i(k_E r - \frac{\pi}{4})}}{\sqrt{r}}$ and find the relation between $f(\varphi)$ and the T matrix.
(4) Use the optical theorem to find $\sigma_{total}$

**The solution:**

(1) Following similar steps as taken in the lecture notes:

Completeness:

$$\int |\vec{k}\rangle \frac{d^2k}{(2\pi)^2}\langle\vec{k}| = \hat{1} \tag{1}$$

Volume element in 2D:

$$d^2k = k\frac{dE}{v_E}d\varphi \tag{2}$$

Using the volume element:

$$\int |\vec{k}\rangle \frac{k_E}{v_E}\frac{dEd\varphi}{(2\pi)^2}\langle\vec{k}| \equiv \int |E,\varphi\rangle \frac{dEd\varphi}{(2\pi)}\langle E,\varphi| \tag{3}$$

From the last line we can recognize:

$$|E,\varphi\rangle = \sqrt{\frac{k_E}{2\pi v_E}}|k\rangle \tag{4}$$

(2) In 2D the green function is:

$$G^+(r_0|r) = -i\frac{m}{2}H_0(k_E|r-r_0|) \tag{5}$$

Were $H_0$ is the Hankel function of the first kind

$$H_0 \simeq \sqrt{\frac{2}{\pi k|r-r_0|}}e^{i(k|r-r_0|-\frac{\pi}{4})} \tag{6}$$

$$G^{+}(r_0|r) \simeq -i\frac{m}{\sqrt{2}}\sqrt{\frac{1}{\pi k|r-r_0|}}e^{i(k|r-r_0|-\frac{\pi}{4})} \tag{7}$$

(3) A density normalized plane wave in 2D:

$$\Phi^{k_0}(r) = e^{i\vec{k_0}\cdot\vec{r}} \tag{8}$$

And we write $\Psi(r)$ as a sum of an incident wave and a scattered wave

$$\Psi(r) = \Phi^{k_0}(r) + \int G(r|r_0)^{+} dr_0 \langle r_0|T|k_0\rangle \tag{9}$$

We now use equation(6) and expand $|\vec{r}-\vec{r_0}|$ in Tailor, this is because only the far field is of interest:

$$\Psi(r) \approx \Phi^{k_0}(r) - i\frac{e^{ik_E(|\vec{r}|)}}{\sqrt{|\vec{r}|}}\frac{m}{\sqrt{2\pi k}}\int e^{i\vec{k_\varphi}\cdot\vec{r_0}} dr_0 \langle r_0|T|k_0\rangle \tag{10}$$

Were we have defined

$$\vec{k}_\varphi = k_E \vec{n}_\varphi \tag{11}$$

Therefore from equation(9) we can recognize $f(\varphi)$:

$$f(\varphi) = \frac{-m}{\sqrt{2\pi k}}\langle \vec{k_\varphi}|T|\vec{k_0}\rangle \tag{12}$$

(4) Using the optical theorem:

$$\sum_{\varphi} |\langle E,\varphi|T|E,\varphi_0\rangle|^2 = -2Im[\langle E,\varphi_0|T|E,\varphi_0\rangle] \tag{13}$$

Using equation(4) we get:

$$\int |\langle \vec{k}_\varphi|T|\vec{k_0}\rangle|^2 d\varphi = \frac{-4\pi v_E}{k_E} Im[\langle \vec{k_0}|T|\vec{k_0}\rangle] \tag{14}$$

Using equation(11):

$$\sigma_{total} = \int |f(\varphi)|^2 d\varphi = 2\sqrt{\frac{2\pi}{k}} Im[f_0] \tag{15}$$