E291: Cross section and optical theorem in 2D

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The problem:

In this problem you have to "generalize" the phase shift method to 2D

(1) Define the channel functions: $\chi_n(\varphi)$.
(2) Write down the free wave functions: $\phi^{E,n}(r, \varphi)$. Keep it normalized.
A plane wave e^{ikx} strikes a "spherical"(round) target.
(3) Write down the incident wave as a sum of "spherical" waves, as was defined in (2).
(4) What is the incident flux in channel n?
Given a phase shift δ_n in channel n.
(5) Find an expression for the partial cross section σ_n.
The asimptotic behaviour of the wave function can be expressed by: $\Psi(x) = e^{ikx} + if(\varphi)\frac{e^{i(k_E r - \frac{\pi}{4})}}{\sqrt{r}}$
(6) Write down an expression for the scattering function $f(\varphi)$.
(7) Find the the proportion coefficient in $\sigma_{total} \propto Im[f(0)]$.

Given:

$$e^{ikx} = e^{ikr\cos(\varphi)} = \sum_{n=-\infty}^{\infty} i^n J_n(kr) e^{in\varphi}$$
$$J_n(z) \sim \sqrt{\tfrac{2}{\pi z}} \cos(z - \tfrac{\pi}{4}(2n+1))$$

The solution:

(1) In 2D the spacial function Y_{lm} is reduced to the normalized function $\chi(\varphi) = \frac{1}{\sqrt{2\pi}} e^{in\varphi}$

(2) The free wave function is of the form

$$\phi^{E,n}(r, \varphi) = A_n(E) J_n(k_E r) \chi_n(\varphi)$$

the ingoing flux is

$$flux = v_E |\phi_{ingoing}|^2$$

using the given approximation, we have (for $r \gg 1$)

$$\phi^{E,n}(r, \varphi) = A_n(E) \sqrt{\frac{2}{\pi k_E r}} \cos(k_E r - \frac{\pi}{4}(2n+1)) \chi_n(\varphi)$$

which helps us recognize the ingoing waves

$$\phi^{E,n}(r, \varphi)_{ingoing} = A_n(E) \sqrt{\frac{1}{2\pi k_E r}} \exp[-i(k_E r - \frac{\pi}{4}(2n+1))] \chi_n(\varphi)$$

now we demand that the total incoming flux (even for a very large r, hence the approxmation) will be 1

$$1 = \int_0^{2\pi} v_E |\phi_{ingoing}|^2 r d\varphi = \frac{v_E}{k_E} (A_n(E))^2$$

hence

$$A_n(E) = \sqrt{\frac{k_E}{v_E}}$$

(3) We shell put the spherical wave function in the given plane wave:

$$e^{ikx} = \sum_{n=-\infty}^{\infty} i^n J_n(kr) e^{in\varphi} = \sum_{n=-\infty}^{\infty} i^n \sqrt{\frac{v_E}{k_E}} \phi^{E,n}(r,\varphi)$$

(4) The incident flux in channel n is the square of the amplitude of the plane wave n term:

$$i_{incident} = |A_n|^2 = \frac{v_E}{k_E}$$

(5) In analogy to the T matrix ($\Psi = \phi + \psi_{scatt}$) we will find ψ_{scatt} from the S matrix. We know that the ingoing and outgoing amplitudes are:

$$A_n = i^n \sqrt{\frac{v_E}{K_E}} \qquad\qquad B_n = S_{nn'} A_n$$

Using the relation: $S = 1 - iT$ and the fact that for a spherical symmetrical target the S matrix is diagonal, we get $T_{nn'} = -\delta_{nn'} e^{i\delta_n} 2\sin(\delta_n)$. We can now derive the outgoing flux:

$$i_n = |-iT_{nn'}A_n|^2 = |\delta_{nn'} e^{i\delta_n} 2\sin(\delta_n) i^{n+1} \sqrt{\frac{v_E}{k_E}}|^2 = 4\sin^2(\delta_n)\frac{v_E}{k_E} \equiv \sigma_n v_E$$

Hence: $\sigma_n = \frac{4}{k_E}\sin^2(\delta_n)$

(6) We use again the same method to find ψ_{scatt} only now we will use the given terms of $J_n(z)$ and e^{ikx}:

$$J_n(kr) \sim \sqrt{\frac{1}{2\pi kr}} e^{i(kr - \frac{\pi}{4}(2n+1))} = \sqrt{\frac{1}{2\pi kr}} e^{i(kr-\frac{\pi}{4})} e^{-i\frac{\pi}{2}n}$$

$$\psi_{scatt} = -iT_{nn'}e^{ikx} = -i\sum_{n=-\infty}^{\infty} (e^{i2\delta_n} - 1) i^n e^{-\frac{\pi}{2}n} \sqrt{\frac{1}{2\pi k}} \frac{e^{i(kr-\frac{\pi}{4})}}{\sqrt{r}} e^{in\varphi} = if(\varphi)\frac{e^{i(kr-\frac{\pi}{4})}}{\sqrt{r}}$$

Hence: $f(\varphi) = \sum_{n=-\infty}^{\infty} \frac{-i}{\sqrt{2\pi k}} (e^{i2\delta_n} - 1) e^{in\varphi}$

(7) We shall use the above result for $f(\varphi)$:

$$|f(\varphi)|^2 = \sum_{n=-\infty}^{\infty} \frac{-i}{\sqrt{2\pi k}} (e^{i2\delta_n} - 1) e^{in\varphi} \sum_{n'=-\infty}^{\infty} \frac{-i}{\sqrt{2\pi k}} (e^{i2\delta'_n} - 1) e^{in'\varphi}$$

$$= \frac{1}{2\pi k} \sum_{nn'} [e^{2i(\delta_n - \delta'_n)} - e^{-2i\delta_n} - e^{2i\delta'_n} + 1] e^{i(n-n')\varphi}$$

$$\sigma_{total} = \int_0^{2\pi} |f(\varphi)|^2 d\varphi = \frac{1}{2\pi k} \sum_{nn'} [e^{2i(\delta_n - \delta'_n)} - e^{-2i\delta_n} - e^{2i\delta_{n'}} + 1] 2\pi\delta(n-n') = \frac{2}{k}\sum_n [1 - \cos(2\delta_n)]$$

Now let's calculate $Im[f(0)]$:

$$Im[f(0)] = Im[\sum_{n=-\infty}^{\infty} \frac{-i}{\sqrt{2\pi k}}(e^{i2\delta_n} - 1)] = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi k}}[1 - \cos(2\delta_n)]$$

We can see that:

$$\sigma_{total} = 2\sqrt{\frac{2\pi}{k}} Im[f(0)]$$

We can derive this result using the optical theorem:

$$\sum_{\varphi} |\langle \phi^{E,\Omega}|T|\phi^{E,\varphi_0}\rangle|^2 = -2Im\langle \phi^{E,\varphi_0}|T|\phi^{E,\varphi_0}\rangle$$

Since for 2D $|E,\varphi\rangle = \sqrt{\frac{k_E}{v_E}}|k\rangle$ and $G(r|r_0) = -i\frac{m}{2}H_0(kr)$, we can calculate the total cross section to obtain the same result.