Ex289: Scattering from a spherical shell

Submitted by: Ori Noked

The problem:
A particle with mass M is scattered from a spherical shell with radius R.
The shell is described by the potential $U(r) = V\delta(r - R)$.
Assume low energy E scattering, so that only s-scattering is important.
You may use the notation $k = (2ME)^{1/2}$ (note that in the class it was denoted k_E).
(1) Find a formula for the phase shift δ.
(2) Write an expression for $\delta^\infty = \delta(V \to \infty)$.
(3) Schematically draw $\delta - \delta^\infty$ as a function of V.
(4) Write an expression for $tan(\delta - \delta^\infty)$.
The resonance energy is defined by the equation $tan(\delta - \delta^\infty) = \infty$.
(5) Find a closed expression for the resonances' width.
The Gamov formula states that the life time is determined by the time period of the particle oscillations inside the well doubled by the transmission factor of the barrier.
(6) Write an expression for the resonance width according to the Gamow formula.
(7) What is the condition for which the result in section (5) will coincide with the Gamow formula.

The solution:

`There is a second version of the solution in the pool`

(1) Because we are solving an $l = 0$ case we are in a semi 1D problem.
We can use $u(r) = rR(r)$(see lecture notes 46.5):
In this case the logarithmic derivative at the boundary of the scattering region is:

$$k_0 = \left(\frac{1}{u(r)}\frac{du(r)}{dr}\right)_r = R \tag{1}$$

We define the scattering region just outside of the well (the delta) and so the matching is:

$$\left(\frac{u'}{u}\right)\Big|_{outside} - \frac{u'}{u}\Big)\Big|_{inside} = 2\alpha \tag{2}$$

Where,

$$\alpha = M\frac{V}{\hbar^2} \tag{3}$$

The outside wavefunction is ingoing free wave and outgoing wave with a phase shift:
For $x < R$:

$$u(r) = A\mathrm{e}^{-ikr} - B\mathrm{e}^{ikr} = A(\mathrm{e}^{-ikr} - \mathrm{e}^{i2\delta}\mathrm{e}^{ikr}) = C\sin(kr + \delta) \tag{4}$$

The inside wave function is simply:
For $x > R$:

$$u(r) = D\sin(kr) \tag{5}$$

Therefore from equation (1) we get:

$$k\cot(kR + \delta) - k\cot(kR) = 2\alpha \tag{6}$$

From this we extract δ:

$$\delta = -kR + \arctan\left(\frac{k}{2MV/\hbar^2 + k\cot(kR)}\right) \tag{7}$$

(2) For $V \to \infty$ we get $\alpha \to \infty$ and the denominator in the arctg goes to zero. Thus we get:

$$\delta^\infty = -kR \tag{8}$$

This is the solution for hard sphere, as expected.

(3) The expression $\delta - \delta^{\infty}$ is:

$$\delta - \delta^{\infty} = \arctan\left(\frac{k}{2MV/\hbar^2 + k\cot(kR)}\right) \tag{9}$$

(4) The expression for $tan(\delta - \delta^{\infty})$ is given from the last section:

$$\tan(\delta - \delta^{\infty}) = \frac{k}{2\alpha + k\cot(kR)} \tag{10}$$

(5) We will define the denominator as $f(E) = 2\alpha + k\cot(kR)$.
We are interested in the energies E_r for which $f(E) = 0$. We will develop to first order:

$$f(E) \approx f(E_r) + \left.\frac{df}{dE}\right|_{E=E_r}(E - Er) \tag{11}$$

When, $f(E_r) = 0$ by definition. We get:

$$f(E) \approx \left.\frac{df}{dE}\right|_{E=E_r}(E - Er) \tag{12}$$

The equation becomes:

$$\left.\frac{df}{dE}\right|_{E=E_r} = \frac{dk}{dE}\cot(kR) - kR\frac{dk}{dE}\frac{1}{\sin^2(kR)} \tag{13}$$

Where,

$$(1) \qquad \frac{dk}{dE} = \frac{M}{\hbar k} = \frac{1}{v_E} \qquad\qquad (2) \qquad \frac{1}{\sin^2(kR)} = 1 + \cot^2(kR) \tag{14}$$

We get:

$$\left.\frac{df}{dE}\right|_{E=E_r} = \frac{1}{v_E}\left(\cot(kR) - kR\left(1 + \cot^2(kR)\right)\right) \tag{15}$$

We remember that:

$$\cot(kR) = -\frac{2\alpha}{k} \tag{16}$$

We get:

$$\left.\frac{df}{dE}\right|_{E=E_r} = -\frac{1}{v_E}\left[\frac{2\alpha}{k} + kR\left(1 + \left(\frac{2\alpha}{k}\right)^2\right)\right] \tag{17}$$

We finally get the familiar form (see lecture notes 46.8):

$$\tan(\delta - \delta^{\infty}) = -\frac{\Gamma_r/2}{E - E_r} \tag{18}$$

Where

$$\Gamma_r = 2\frac{v_E}{R}\left[1 + \frac{(2\alpha)/R + (2\alpha)^2}{k^2}\right]^{-1} = \frac{v_E}{2R}\cdot 4\left[1 + \frac{(2\alpha)/R + (2\alpha)^2}{k^2}\right]^{-1} \tag{19}$$

We will note that for high barrier $V \to \infty$ we will get $\alpha \to \infty$.
If we will assume that k and R do not change, we get:

$$(1) \qquad \alpha >> 1/R \qquad\qquad (2) \qquad \alpha >> k \tag{20}$$

$$\Gamma_r \approx 2\frac{v_E}{R}\left(\frac{k}{2\alpha}\right)^2 = \frac{v_E}{2R}\left(\frac{k}{\alpha}\right)^2 \tag{21}$$

(6) The Gamow formula expression for the resonance width is (see lecture notes 21.3-4):

$$\Gamma_r = f_a \cdot g \tag{22}$$

Where we define the "attempt frequency" f_a and the transmission g as:

$$(1) \quad f_a = v_E/2R \qquad (2) \quad g = \frac{1}{1+(\alpha/k)^2} \approx \left(\frac{k}{\alpha}\right)^2 \tag{23}$$

We get:

$$\Gamma_r = \frac{v_E}{2R}\left(\frac{k}{\alpha}\right)^2 \tag{24}$$

Another way to define Γ_r is:

$$\Gamma_r = 2v_r\gamma_r \tag{25}$$

Where: $v_r = v_E$ is the velocity of the particle and γ_r is the imaginary part of the momentum.

$$(1) \quad v_E = \sqrt{2E/M} \qquad (2) \quad \gamma_r = \frac{1}{R}\left(\frac{k}{2\alpha}\right)^2 \tag{26}$$

(7) We can see from the comparison of the two Γ_r that the condition is that the barrier is very high.
In this case there are two approximations:
First, the hight of the barrier comparing to the well size: $(2\alpha)/R << (2\alpha)^2 \rightarrow \alpha >> 1/R$

$$\left[1 + \frac{(2\alpha)/R + (2\alpha)^2}{k^2}\right]^{-1} \approx \left[1 + \left(\frac{2\alpha}{k}\right)^2\right]^{-1} \tag{27}$$

Second, the hight of the barrier comparing to the momentum of the particle: $\alpha >> k$

$$\textit{Scattering} \quad \left[1 + \left(\frac{2\alpha}{k}\right)^2\right]^{-1} \approx \left(\frac{k}{2\alpha}\right)^2 \quad \texttt{and} \quad \frac{1}{1+(\alpha/k)^2} \approx \left(\frac{k}{\alpha}\right)^2 \quad \textit{Gamow} \tag{28}$$

We get the "attempt frequency":

$$f_a = \frac{v_E}{2R} \tag{29}$$

And the transmission g is:

$$g(\textit{Scattering}) = 4\left[1 + \frac{(2\alpha)/R + (2\alpha)^2}{k^2}\right]^{-1} \approx 4\left[1 + \left(\frac{2\alpha}{k}\right)^2\right]^{-1} \approx \underbrace{\left(\frac{k}{\alpha}\right)^2}_{\textit{final formula}} \approx \frac{1}{1+(\alpha/k)^2} = g(\textit{Gamow}) \tag{30}$$

The final formula is the same:

$$\Gamma_r = f_a \cdot g = \frac{v_E}{2R}\left(\frac{k}{\alpha}\right)^2 \tag{31}$$