E287: scattering from a spherical shell, phase shifts

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The problem:

find the phase shifts δ_l caused as a result of scattering from a sphere of radius a ,with a large but not infinite potential V_0, and that is within a spherical shell potential of the form $V(r) = u_0\delta(r-a)$. the sphere is a small one, so that $k_E a \ll 1$, $\alpha a \ll 1$.

The solution:

the non- negligible scattering is within the channels that satisfy $\frac{l^2}{2ma^2} < E$, we are left with simply $l = 0$ because the condition becomes $l < k_E a \Rightarrow l \ll 1$. however, for practice, we'll solve for any l as well.

starting with $l = 0$:

for $0 < r < a$ the logarithmic derivative is:

$$\tilde{k_0} = (\frac{1}{u(r)}\frac{d(u(r))}{dr}_{r=a}) = \alpha\coth(\alpha a) + 2mu_0 \approx \frac{1}{a} + 2mu_0$$

$$\alpha = \sqrt{2m(V_0 - E)} \approx \sqrt{2mV_0}$$

the solution for $r > a$ is:

$$u(r) = C(\cos(\delta_0)\sin(k_E r) + \sin(\delta_0)\cos(k_E r)) = C\sin(\delta_0 + k_E r)$$

$$k_E = \sqrt{2mE}$$

from the matching eq. we get:

$$k_E\cot(\delta_0 + k_E a) = \tilde{k_0}$$

and the phase shift:

$$\delta_0 = -k_E a + \arctan(\frac{k_E}{\tilde{k_0}}) \approx -k_E a + \arctan(\frac{k_E a}{1 + 2mu_0 a})$$

now for any l:

inside we have:

$$k_l = (\frac{1}{R}\frac{dR}{dr}_{r=a}) = \alpha\frac{j_l'(i\alpha a)}{j_l(i\alpha a)} + 2mu_0$$

$$(k_0 = \tilde{k_0} - \frac{1}{a})$$

and outside:

$$R(r) = C(\cos(\delta_l)j_l(k_E r) + \sin(\delta_l)n_l(k_E r))$$

the matching eq. gives:

$$\tan(\delta_l) = -\frac{k_l j_l(k_E a) - k_E j_l'(k_E a)}{k_l n_l(k_E a) - k_E n_l'(k_E a)}$$

for $x \ll 1$, $(l \neq 0)$ the bessel func. get the form:

$$j_l(x) \rightarrow \frac{(x)^l}{(2l+1)!!}$$

$$n_l(x) \rightarrow \frac{(2l-1)!!}{(x)^{l+1}}$$

$$j_l'(x) \rightarrow l\frac{(x)^{l-1}}{(2l+1)!!}$$

$$n_l'(x) \rightarrow -(l+1)\frac{(2l-1)!!}{(x)^{l+2}}$$

using $k_E a \ll 1, \alpha a \ll 1$ we get:

$$\tan(\delta_l) = -\frac{k_l \frac{(k_E a)^l}{(2l+1)!!} - k_E l \frac{(k_E a)^{l-1}}{(2l+1)!!}}{k_E(l+1)\frac{(2l-1)!!}{(k_E a)^{l+2}} + k_l \frac{(2l-1)!!}{(k_E a)^{l+1}}} = -\frac{(k_E a)^{2l+1}}{(2l+1)!!(2l-1)!!}\frac{k_l - \frac{l}{a}}{k_l + \frac{l+1}{a}}$$

$$k_l = i\alpha \frac{l\frac{(i\alpha a)^{l-1}}{(2l+1)!!}}{\frac{(i\alpha a)^l}{(2l+1)!!}} + 2mu_0 = \frac{l}{a} + 2mu_0$$

and the phase shift becomes:

$$\delta_l = -\arctan\left(\frac{(k_E a)^{2l+1}}{(2l+1)!!(2l-1)!!}\frac{2mu_0}{2mu_0 + \frac{2l+1}{a}}\right)$$

the units make sense since $[k_E a] = 1$ $[mu_0] = [\frac{1}{a}]$.
we can also see $\forall l \neq 0 : \delta_l \ll \delta_0$ and that V_0 falls in this approximation.
trying to simplify using $x \ll 1 \rightarrow \tan(x) \approx \sin(x) \approx x$

gives this final result:

$$\delta_l \approx -\frac{(k_E a)^{2l+1}}{(2l+1)!!(2l-1)!!}\frac{2mu_0 a}{2mu_0 a + 2l + 1}$$

a comparison between the δ_0 expressions:
the expression in the case of $l = 0$ becomes: $\delta_{l=0} \approx -k_E a \frac{2mu_0 a}{2mu_0 a + 1}$
and using the $\tan(x) \approx x$ approximation again, for the l=0 expression gives the same answer: $\delta_0 \approx -k_E a(\frac{2mu_0 a}{1+2mu_0 a})$