E285: Scattering from $\frac{1}{r^2}$ potential

Submitted by: Anat Dulman

The problem:

(1) Find the phase shifts caused by scattering from $\frac{a}{r^2}$ potential.

(2) Calculate the deferential cross section for small a.

The solution:

(1)

$$\mathcal{H}^{(l)} = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - [l(l+1) + 2ma]\frac{1}{r^2}$$

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k_E^2 - \frac{l^{eff}(l^{eff}+1)}{r^2}\right] R^l(r) = 0$$

Where

$$l^{eff} = -\frac{1}{2} + \sqrt{\frac{1}{4} + l(l+1) + 2ma} = -\frac{1}{2} + \sqrt{(l+\frac{1}{2})^2 + 2ma}$$

from the last equality we can see that when $a = 0$ we get $l^{eff} = l$
When $a = 0$ the solution ($r \to \infty$) is a spherical free wave -

$$R^l(r) \sim \frac{k_E}{\sqrt{v_E}} \frac{\sin\left(k_E r - \frac{\pi l}{2}\right)}{k_E r}$$

When $a \neq 0$ the solution ($r \to \infty$) is

$$R^l(r) \sim \frac{k_E}{\sqrt{v_E}} \frac{\sin\left(k_E r - \frac{1}{2}\pi l^{eff}\right)}{k_E r} = \frac{k_E}{\sqrt{v_E}} \frac{\sin\left(k_E r - \frac{\pi l}{2} + \delta_l\right)}{k_E r}$$

and $\delta_l = \frac{\pi}{2}(l - l^{eff})$.

(2) For $a \ll 1$:

$$l^{eff} = -\frac{1}{2} + (l+\frac{1}{2})\sqrt{1 + \frac{2ma}{(l+\frac{1}{2})^2}} \approx l + \frac{ma}{(l+\frac{1}{2})}$$

and

$$\delta_l = -\frac{\pi}{2}\frac{ma}{(l+\frac{1}{2})}$$

As mentioned in the lecture notes:

$$f(\Omega) = -\frac{1}{k_E}\sum_l (l+\frac{1}{2}) T_{ll} P_l(cos\theta)$$

where $T_{ll} = e^{i\delta_l}\sin\delta_l$.

Substituting T_{ll} and expanding for small δ:

$$f(\Omega) = \frac{1}{k_E}\sum_l (l+\frac{1}{2})(\frac{\pi}{2}\frac{ma}{(l+\frac{1}{2})}) P_l(cos\theta) = \frac{\pi}{2}\frac{ma}{k_E}\sum_l P_l(cos\theta)$$

For the summation we will use the generating function of Legendre polynomials (Schaum) :

$$\sum_{n=0}^{\infty} P_n(x)t^n = \frac{1}{\sqrt{1-2tx+t^2}}$$

Using the last equation for $t = 1$ we get:

$$f(\Omega) = \frac{\pi}{2}\frac{ma}{k_E}\frac{1}{\sqrt{2(1-\cos\theta)}} = \frac{\pi}{4}\frac{ma}{k_E\sin\theta/2}$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2 = \frac{\pi^2(ma)^2}{16k_E^2 sin^2(\theta/2)}$$

The total cross section

$$\sigma_{total} = \frac{\pi^3(ma)^2}{8k_E^2}\int_0^{\pi}\frac{sin(\theta)d\theta}{sin^2(\theta/2)} = \frac{\pi^3(ma)^2}{4k_E^2}\int_{-1}^{1}\frac{dcos\theta}{1-cos\theta} = \frac{\pi^3(ma)^2}{4k_E^2}\ln|1-cos\theta||_{-1}^{1} = \infty$$

We can also see that directly from this formula taken from the lecture notes

$$\sigma_{total} = \frac{4\pi}{k_E^2}\sum_l (2l+1)sin^2(\delta_l) \approx \frac{4\pi}{k_E^2}\sum_l (2l+1)\frac{(\pi ma)^2}{(2l+1)^2} = \frac{4\pi^3(ma)^2}{k_E^2}\sum_l \frac{1}{(2l+1)}$$

This is a harmonic series which gives infinity.