E283: Phase shifts in scattering from a delta function

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The problem:

Definition of 1D delta:
$V(x) = u\delta(x)$
means "rectangular" of width a and height U_0 such that $u = U_0 a$, where a is taken to be very small.
similarly a 3D delta function is a sphere of radius a and "height" U_0 such that $u = (4\pi/3)U_0 a^3$, where a is taken to be very small and u is maintained constant. In this question you are requested to calculate the phase shifts in scattering on a 3D delta function $V(r) = u\delta(x)\delta(y)\delta(z)$.

The solution:

first we need to define the regime in which this potential is. the "height", U_0 is a function of u and a: $U_0 = (3/4\pi)ua^{-3}$. it is undoubtedly a small sphere since for every energy E we can choose a sphere radius $a > o$ so that $k_E a \ll 1$. since we can always start with a radius a such that $U_0 > E$ we present a solution for the inner part of the form:

$$\psi\left(r\right)_{r<a} = j_l\left(i\alpha r\right), \alpha = \sqrt{2m\left(U_0 - E\right)} \tag{1}$$

the outer part solution is:

$$\begin{aligned} R^l_{r>a}\left(r\right) &= Ah^-_l\left(k_E r\right) + Bh^+_l\left(k_E r\right) \\ &= Ah^-_l\left(k_E r\right) + e^{i2\delta_l}h^+_l\left(k_E r\right) \\ &= C\left[\cos\left(\delta_l\right)j_l\left(k_E r\right) + \sin\left(\delta_l\right)n_l\left(k_E r\right)\right] \end{aligned} \tag{2}$$

where h_l is the Henkel function and j_l, n_l are the first and second order Bessel functions. the matching condition demands equal logarithmic derivatives on the boundary. we define:

$$k_l = \left.\left(\frac{1}{R_l\left(r\right)}\frac{dR_l\left(r\right)}{dr}\right)\right|_{r=a} \tag{3}$$

for $r < a$.we can see that $\lim\limits_{a\ll 1}\alpha \approx \sqrt{2mU_0} = \sqrt{(3/2\pi)mu}a^{-3/2} \gg 1$ and also $\alpha a \approx \sqrt{(3/2\pi)mu}a^{-1/2} \gg 1$.

it is important to note that this is unique to the 3D case, whereas in the 2D case $\alpha = O\left(a^{-1}\right) \Rightarrow \alpha a = O\left(1\right)$, and in the 1D case $\alpha = O\left(a^{-1/2}\right) \Rightarrow \alpha a = O\left(a^{1/2}\right)$. this leads to completely different behavior in the $r = a$ boundary. back to the 3d: We saw before that this is the small sphere regime. in this regime we have 2 approximations: the hard sphere and Born. to check where we stand we need to look at what happens to k_l in the limit of small a. first we calculate k_l from the interior solution. since $\alpha a \gg 1$ we can use the long range estimate of j_l we saw in class:

$$j_l\left(kr\right) \approx \frac{\sin\left(kr - \frac{\pi}{2}l\right)}{kr}$$

for k_l to get:

$$k_l = \left(\frac{1}{j_l\left(i\alpha r\right)}\frac{d\left(j_l\left(i\alpha r\right)\right)}{dr}\right)_{r=a} = \alpha\coth\left(\alpha a - \frac{\pi}{2}l\right) - \frac{1}{a} \approx \alpha - \frac{1}{a} \approx \sqrt{\frac{3mu}{2\pi a^3}} \underset{a\to 0}{\to} \infty$$

this means we're in the hard sphere limit. we saw in class that:

$$\tan\left(\delta_l\right) = -\frac{k_l j\left(k_E a\right) - k_E j'\left(k_E a\right)}{k_l n\left(k_E a\right) - k_E n'\left(k_E a\right)} \tag{4}$$

using the short range $l \gg k_E r$ estimates:

$$\begin{aligned} n_l\left(r\right) &\approx (2l-1)!!\left(\tfrac{1}{r}\right)^{l+1}\left[1 + \tfrac{1}{2(2l-1)}r^2 + ...\right] \\ j_l\left(r\right) &\approx \tfrac{r^l}{(2l+1)!!}\left[1 - \tfrac{1}{2(2l+3)}r^2 + ...\right] \end{aligned} \tag{5}$$

and $k_l = \infty$ we get:

$$\tan\left(\delta_l^{\infty}\right) \approx -\frac{1}{(2l+1)!!\,(2l-1)!!}\left(k_E a\right)^{2l+1} \tag{6}$$

which goes to 0 when a goes to zero. this is consistent with the born approximation in the short range for $l \neq 0$:

$$\delta_l^{Born}\left(0\right) \approx -\frac{2}{(2l+3)\left[(2l+1)!!\right]^2}\frac{3mu}{4\pi\hbar^2 a^3}a^2\left(k_E a\right)^{2l+1} \tag{7}$$

however, for $l = 0$ the born approximation gives a non vanishing term:

$$\delta_0^{Born}\left(0\right) \approx -\frac{2}{3}\frac{3mu}{4\pi\hbar^2 a^3}a^2\left(k_E a\right) = -\frac{mu}{2\pi\hbar^2}k_E \tag{8}$$

this is a mistake that is caused by the fact that the born approximation isn't valid for $U_0 > \frac{\hbar^2}{ma^2}$. In our case:

$$U_0 = \frac{3u}{4\pi a^3} > \frac{\hbar^2}{ma^2} \Rightarrow a < \frac{3mu}{4\pi\hbar^2}$$

so, the born approximation isn't valid for any smaller radius.

Appendix:

in this exercise i chose to first take the hard sphere limit and then the short range expansion. however, this can be also be done by first taking the short range approximation:

$$\tan\left(\delta_l\right) \approx -\frac{\left((2l+1)!!\right)^{-1}\left(k_l\right)\left(k_E a\right)^l\left[1-\frac{(ka)^2}{2(2l+3)}+...\right]-k_E\left[l\left(k_E a\right)^{l-1}-\frac{(l+1)(k_E a)^{l+1}}{2(2l+3)}\right]}{(2l-1)!!\left[\left(k_l\right)\left(k_E a\right)^{-(l+1)}\left[1+\frac{(ka)^2}{2(2l-1)}\right]-k_E\left[-\left(l+1\right)\left(k_E a\right)^{-(l+2)}-\frac{(l-1)(k_E a)^{-l}}{2(2l-1)}\right]\right]} \tag{9}$$

and then taking the hard sphere limit $k_l \rightarrow \infty$.