Ex2672: Two sites scatterer

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The problem:
Particle A is located in superposition of two sites. The two sites are at distance d from each other. Particle B is a free particle being scattered by particle A. Particle A creates a potential of $u\delta(x_1-x_2)$ where x_1 is the location of particle A and x_2 is the location of particle B.
The Hamiltonian is given in the standard base:

$$H=\frac{p^2}{2m}-\frac{h}{2}\left(\sigma_x+1\right)+u\delta(x-\frac{d}{2}\left(\sigma_z+1\right)) \tag{1}$$

Where

- h is the transfer amplitude between the two particle A sites.
- m is the mass of particle B.
- p is the momentum of particle B.

Assume that particle A is located in a symmetric superposition.
In the following questions it might be useful to change to symmetric and asymmetric basis of the problem.

1. Write the equations defining the transmission coefficient of the scatterer.
2. Solve exactly for the special cases of
   (a) $h>E$
   (b) $h=0$
   (c) $h\to-\infty$
3. Show the numeric solution in the mid-range $0<h\ll\infty$

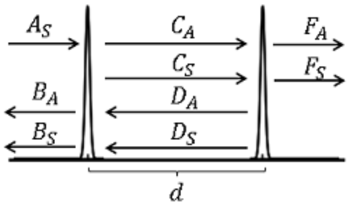
The solution:
On a first look of the Hamiltonian we can see that any quantum state will be defined by two separable parameters: $|k,n\rangle$
while $|k\rangle$is the momentum of the free particle. $|n\rangle\in\{|0\rangle,|1\rangle\}$ is the position of the barrier. This would be our standard basis.
For every X except $X=0,d$ the Hamiltonian is diagonal at the symmetric and anti-symmetric basis of the barrier.
Therefore, it is convenient to change to a basis of the n states as follows:

$$\begin{cases} |S\rangle=\frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right) \\ |A\rangle=\frac{1}{\sqrt{2}}\left(|0\rangle-|1\rangle\right) \end{cases} \tag{2}$$

And the Hamiltonian becomes:

$$H=\frac{p^2}{2m}-\frac{h}{2}\left(\sigma_z+1\right)+u\delta(x-\frac{d}{2}\left(\sigma_x+1\right)) \tag{3}$$

$|S\rangle$ and $|A\rangle$ stands for the symmetric and anti-symmetric states respectfully.
In this new basis the momentum eigenvalues are $k_S=\sqrt{2mE}$ and $k_A=\sqrt{2m\left(E-h\right)}$.
We define the amplitudes of the wavefunctions for each area as follows:



Setting the incident wave amplitude to be only on the symmetric channel we set:

$$\begin{cases} A_S = 1 \\ A_A = 0 \end{cases} \tag{4}$$

Therefore, the wave functions in the different areas can be written as:

$$\begin{pmatrix} \psi_S \\ \psi_A \end{pmatrix} = \begin{cases} \begin{pmatrix} e^{ik_Sx} + B_Se^{-ik_Sx} \\ B_Ae^{-ik_Ax} \end{pmatrix} & x \leq 0 \\ \begin{pmatrix} C_Se^{ik_Sx} + D_Se^{-ik_Sx} \\ C_Ae^{ik_Ax} + D_Ae^{-ik_Ax} \end{pmatrix} & 0 \leq x \leq d \\ \begin{pmatrix} F_Se^{ik_Sx} \\ F_Ae^{ik_Ax} \end{pmatrix} & x \geq d \end{cases} \tag{5}$$

1.

The boundary conditions for the wave-functions are:

$$\begin{cases} \psi\left(x = 0^+\right) = \psi\left(x = 0^-\right) \\ \psi\left(x = d^+\right) = \psi\left(x = d^-\right) \end{cases} \tag{6}$$

It is easier to write the derivative equations in the position basis. Hence, we'll revert back to this basis:

$$\begin{cases} \psi_0 = \frac{1}{\sqrt{2}}\left(\psi_A + \psi_S\right) \\ \psi_1 = \frac{1}{\sqrt{2}}\left(\psi_A - \psi_S\right) \end{cases} \tag{7}$$

The boundary conditions for the wave-functions derivatives are:

$$\begin{cases} \frac{d\psi_0}{dx}\left(x = 0^+\right) - \frac{d\psi_0}{dx}\left(x = 0^-\right) = 2mu\psi_0\left(x = 0\right) \\ \frac{d\psi_1}{dx}\left(x = 0^+\right) - \frac{d\psi_1}{dx}\left(x = 0^-\right) = 0 \\ \frac{d\psi_0}{dx}\left(x = d^+\right) - \frac{d\psi_0}{dx}\left(x = d^-\right) = 0 \\ \frac{d\psi_1}{dx}\left(x = d^+\right) - \frac{d\psi_1}{dx}\left(x = d^-\right) = 2mu\psi_1\left(x = d\right) \end{cases} \tag{8}$$

Substituting the wave functions (5) into the boundary conditions (6), (8), we get the following 8 equations:

At $x = 0$

$1 + B_S = C_S + D_S$
$B_A = C_A + D_A$
$\left[k_SC_S - k_SD_S + k_AC_A - k_AD_A - \left(k_S - k_SB_S - k_AB_A\right)\right] i = 2mu\left(1 + B_S + B_A\right)$
$k_SC_S - k_SD_S - k_AC_A + k_AD_A - \left(k_S - k_SB_S + k_AB_A\right) = 0$

At $x = d$

$C_S e^{ik_S d} + D_S e^{-ik_S d} = F_S e^{ik_S d}$
$C_A e^{ik_A d} + D_A e^{-ik_A d} = F_A e^{ik_A d}$
$F_S k_S e^{ik_S d} + F_A k_A e^{ik_A d} - \left(k_S C_S e^{ik_S d} - k_S D_S e^{-ik_S d} + C_A k_A e^{ik_A d} - k_A D_A e^{-ik_A d}\right) = 0$
$\left[F_S k_S e^{ik_S d} - F_A k_A e^{ik_A d} - \left(k_S C_S e^{ik_S d} - k_S D_S e^{-ik_S d} - \left(C_A k_A e^{ik_A d} - k_A D_A e^{-ik_A d}\right)\right)\right] i = 2mu \left(F_S e^{ik_S d} - F_A e^{ik_A d}\right)$

2.

An impotrant comment about probability calculation

Calculating the transmission and refelection probabilities we usually ignore the dependence of those factors on the velocity:

$$F_A e^{ik_A x} = \frac{\tilde{F}_A}{\sqrt{v_A}} e^{ik_A x} \tag{9}$$

We usually igonre this fact when handling scattering problems because the velocity is the same for every wave. In this case, we should go back and remember the dispersion relation:

$$v = \frac{k}{m} \tag{10}$$

Therefore an amplitude in one channel that is depended on an amplitude of another channel using each channel k value. For example, calculating the outgoing flux in the anti-symmetric channel:

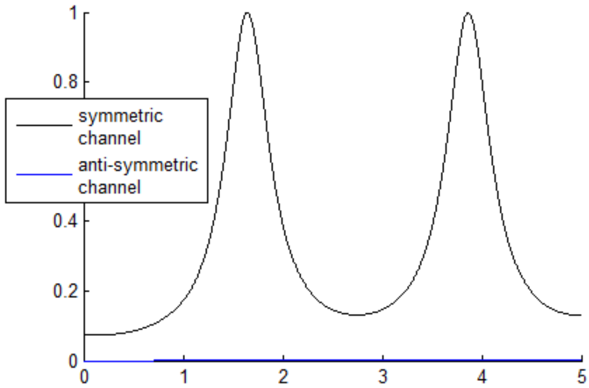
$$\tilde{F}_A = t_A \tilde{A}_S \tag{11}$$

We must write:

$$t_A = \sqrt{\frac{k_A}{k_S}} \left(\frac{F_A}{A_S}\right) \tag{12}$$

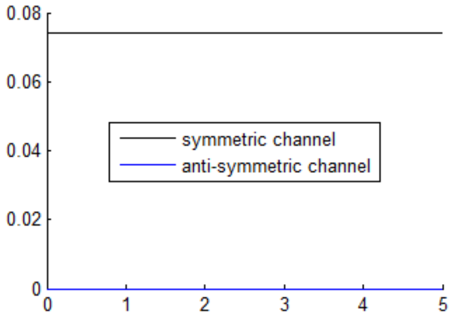
a. The case of $h > E$

Easy to see that for $h > E$ we get $k_A = i\rho$ and the particle wave functions at the anti-symmetric channel becomes decaying exponents. Looking far from the scatterer we have $B_A = F_A = 0$ and when $d \gg \frac{1}{\rho}$ the C_A, D_A are neglectables.



b. The case of $h = 0$

Under this terms we can see that σ_x will be a constant of motion.
Looking only on state$|1\rangle$ (without lost of generality) the problem is the known problem of a single delta scatterer at the respected location ($x = 0$ or $x = d$).
The transmission and reflection coefficients of a delta scatterer are $t = \frac{ik}{ik-mu}$ and $r = \frac{mu}{ik-mu}$.
Applying them on our incident wave $\psi_S = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ we get the transmitted state:
$\psi_t = \frac{1}{\sqrt{2}}(t\,|0\rangle + t\,|1\rangle) = t\psi_S$
The reflecteld state is:
$\psi_r = \frac{1}{\sqrt{2}}(r\,|0\rangle + re^{i2k_Sd}\,|1\rangle) = \frac{re^{ik_Sd}}{\sqrt{2}}(e^{-ik_Sd}\,|0\rangle + e^{ik_Sd}\,|1\rangle) = re^{ik_Sd}[cos(k_Sd)\psi_S - isin(k_Sd)\psi_A]$
In other words we get$|F_S|^2 = |t|^2$, $|F_A|^2 = 0$, $|B_A|^2 = |r \cdot cos(k_Ad)|^2$ and $|B_S|^2 = |r \cdot sin(k_Ad)|^2$

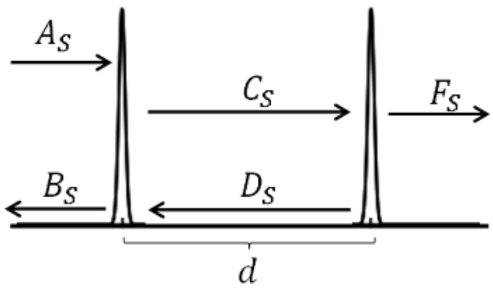


c. The case of $h \to -\infty$

In this case $k_A = \sqrt{2m\,(E-h)} \to \infty$. For that reason, we can neglet k_S and constants from our boundry equations. The yielded set of equation is of A_A, B_A, C_A and F_A only (without A_S, B_S, C_S and F_S or any constants). Since we set $A_A = 0$ we have the trivial solution:

$$B_A = C_A = F_A = 0 \tag{13}$$

In other words, there is no wavefuntion in the anti-symmetric channel.
This case is identical to Febri-Perrot. We shall explain.
One can substitute $B_A = C_A = F_A = 0$ into the boundary conditions and have:
$1 + B_S = C_S + D_S$
$[k_SC_S - k_SD_S - (k_S - k_SB_S)]\,i = mu\,(1 + B_S)$
$C_Se^{ik_Sd} + D_Se^{-ik_Sd} = F_Se^{ik_Sd}$
$\left[F_Sk_Se^{ik_Sd} - \left(k_SC_Se^{ik_Sd} - k_SD_Se^{-ik_Sd}\right)\right] i = mu\left(F_Se^{ik_Sd}\right)$
Which is precisely the set of equation for the Febri-Perrot problem in case that the scattering amplitude is $u \to \frac{u}{2}$:



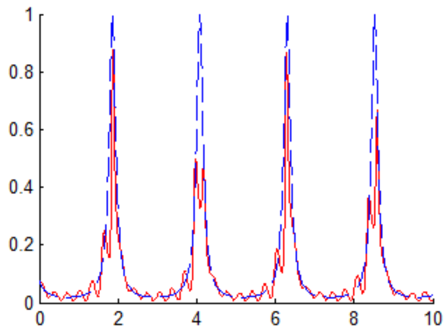
Comparsion of $h \to -\infty$ and the Febri-Perrot analysis

From the lecturer notes, the transmission of Febri-Perrot scatterer is

$$transmission = \frac{1}{1 + 4\left(R/T^2\right)\left(\sin\left(\phi\right)\right)^2} \tag{14}$$

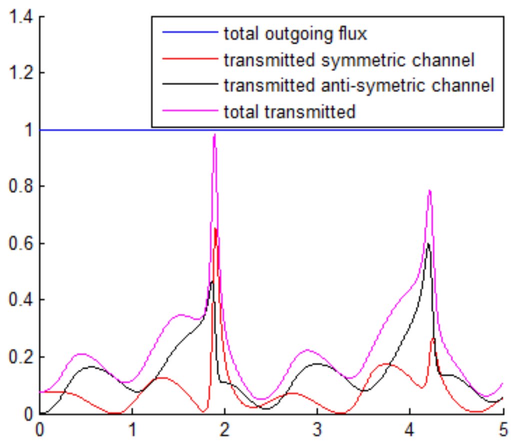
Details in lecture notes under "The two-terminal delta junction" in the "Boxes and Networks" chapters. Where $R = |r|^2$ and $T = |t|^2$.
Plotted below are:

- Red - the result for the general case, taking very large h
- Blue - the result of the equivalent Febri-Perrot problem



3. The case of mid-range $0 < h \ll \infty$

We solve this system of equations for given E, h, u, d in MATLAB.
The transmission coefficients are denoted F_S^2, F_A^2.
Plotting the numeric results for $|F_A|^2$ and $|F_S|^2$ for different h values will give us good intuition of the system behavior.



As we would have expected, the total outgoing flux is 1:

$$J_{in} = J_{out} \Rightarrow |F_A|^2 + |F_S|^2 + |B_A|^2 + |B_S|^2 = 1 \tag{15}$$

For any mid-range h we can see two periodicities with a different frequency. The high frequency equivalent to the case of $h \to \infty$ with the same other values (k, u). The low frequency relied on $\Delta k = k_S - k_A = \sqrt{2mE} - \sqrt{2m(E-2h)}$. In our case the variety of paths not only depends on the number of recurrences but also on the channel. In each channel k is different therefore the phase accumulated.
As a rule of thumb, an increase in $\frac{E}{u}$ cause increase of $\frac{T}{R}$ (where $T = |F_A|^2 + |F_S|^2$ the transmission amplitude and $R = |B_A|^2 + |B_S|^2$ the reflection amplitude). While any change in E (therefor a change in k) will involve in a change of the periodicy dependence of the tansition amplitude on d.
A simple check can assure $T + R = 1$ at any configuration.