E259: Scattering by harmonic oscillator

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The problem:

The solution below is for scattering on $V(x) = Q\delta(x)$ where $Q = \sum_\alpha c_\alpha Q_\alpha$. The case of a single oscillator is a special case. Similar strategy can be used to solve problem E258 of scattering by a delta in a waveguide. In the latter case $Q = c\delta(y - y_0)$.

The solution:

The Hamiltonian of the particle plus the local bath is

$$\mathcal{H} = \frac{p^2}{2\mathsf{m}} + \delta(x)\sum_\alpha c_\alpha Q_\alpha + \sum_\alpha \left(\frac{{P_\alpha}^2}{2\mathsf{m}_\alpha} + \frac{1}{2}\mathsf{m}_\alpha\omega_\alpha^2 {Q_\alpha}^2\right) \tag{1}$$

From now on we use the notation

$$\hat{Q} = \sum_\alpha c_\alpha \hat{Q}_\alpha \tag{2}$$

Outside of the scattering region the total energy of the system (particle plus bath minus floor) is

$$\mathcal{E} \;=\; \epsilon + E_n \;=\; \epsilon + \sum_\alpha n_\alpha \omega_\alpha \tag{3}$$

We look for scattering states that satisfy the equation

$$\mathcal{H}|\Psi\rangle = \mathcal{E}|\Psi\rangle$$

The scattering channels are labeled as

$$\boldsymbol{n} = (n_0, n) = (n_0; n_1, n_2, n_3, ..., n_\alpha, ...) \tag{4}$$

where $n_0 = \text{left,right}$. We define

$$k_{\boldsymbol{n}} = \sqrt{2m(\mathcal{E} - E_n)} \qquad \text{for } n \in \text{open} \tag{5}$$

$$\alpha_{\boldsymbol{n}} = \sqrt{-2m(\mathcal{E} - E_n)} \qquad \text{for } n \in \text{closed} \tag{6}$$

later we use the notations

$$v_{\boldsymbol{n}} = k_{\boldsymbol{n}}/\mathsf{m} \tag{7}$$

$$u_{\boldsymbol{n}} = \alpha_{\boldsymbol{n}}/\mathsf{m} \tag{8}$$

and define diagonal matrices $\boldsymbol{v} = \text{diag}\{v_n\}$ and $\boldsymbol{u} = \text{diag}\{u_n\}$. The channel radial functions are written as

$$R(r) = A_n e^{-ik_n r} + B_n e^{+ik_n r} \qquad \text{for } n \in \text{open} \tag{9}$$

$$R(r) = C_n e^{-\alpha_n r} \qquad \text{for } n \in \text{closed} \tag{10}$$

Next we derive expression for the $2N \times 2N$ matching matrix $\boldsymbol{M}$ and for the $2N \times 2N$ scattering matrix $\boldsymbol{S}$, where N is the number of open modes.

The wavefunction can be written as

$$\Psi(r, n_0, Q) = \sum_{n_1, n_2, ...} R_{n_0, n_1, n_2, ...}(r)\chi^{n_1, n_2, \cdots}(Q) \tag{11}$$

The matching equations are

$$\Psi(0, \text{right}, Q) = \Psi(0, \text{left}, Q) \tag{12}$$

$$\frac{{}^{2}}{2\mathsf{m}}\left[\Psi'(0, \text{right}, Q) + \Psi'(0, \text{left}, Q)\right] = \hat{Q}\Psi(0, Q) \tag{13}$$

The operator $\hat{Q}$ is represented by the matrix Q_{nm} that has the block structure

$$Q_{nm} = \begin{pmatrix} Q_{vv} & Q_{vu} \\ Q_{uv} & Q_{uu} \end{pmatrix} \tag{14}$$

For sake of later use we define

$$M_{nm} = \begin{pmatrix} \frac{1}{\sqrt{v}} Q_{vv} \frac{1}{\sqrt{v}} & \frac{1}{\sqrt{v}} Q_{vu} \frac{1}{\sqrt{u}} \\ \frac{1}{\sqrt{u}} Q_{uv} \frac{1}{\sqrt{v}} & \frac{1}{\sqrt{u}} Q_{uu} \frac{1}{\sqrt{u}} \end{pmatrix} \tag{15}$$

The matching conditions lead to the following set of matrix equations

$$\begin{aligned} A_R + B_R &= A_L + B_L && (16) \\ C_R &= C_L && (17) \\ -i\boldsymbol{v}(A_R - B_R + A_L - B_L) &= 2Q_{vv}(A_L + B_L) + 2Q_{vu}C_L && (18) \\ -\boldsymbol{u}(C_R + C_L) &= 2Q_{uv}(A_L + B_L) + 2Q_{uu}C_L && (19) \end{aligned}$$

from here we get

$$\begin{aligned} A_R + B_R &= A_L + B_L && (20) \\ A_R - B_R + A_L - B_L &= i2(\boldsymbol{v})^{-1}\mathcal{Q}(A_L + B_L) && (21) \end{aligned}$$

where

$$\mathcal{Q} = Q_{vv} - Q_{vu} \frac{1}{(\boldsymbol{u} + Q_{uu})} Q_{uv} \tag{22}$$

The set of matching conditions can be written as

$$\begin{pmatrix} B_R \\ A_R \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} A_L \\ B_L \end{pmatrix} \tag{23}$$

with the matching matrix

$$\boldsymbol{M} = \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{pmatrix} = \begin{pmatrix} 1 - i\mathcal{M} & -i\mathcal{M} \\ i\mathcal{M} & 1 + i\mathcal{M} \end{pmatrix} \tag{24}$$

where $\mathcal{M} = (\boldsymbol{v})^{-1}\mathcal{Q}$.

We want to switch to flux normalization convention, so as to get a unitary $\boldsymbol{S}$ matrix. In this convention the proper definition of the amplitudes is $\tilde{A_n} = \sqrt{v_n} A_n$ and $\tilde{B_n} = \sqrt{v_n} B_n$. Therefore we redefine the $\mathcal{M}$ matrix using the prescription

$$\mathcal{M}_{nm} \mapsto \sqrt{v_n} \mathcal{M}_{nm} \frac{1}{\sqrt{v_m}} \tag{25}$$

leading to

$$\mathcal{M} = \frac{1}{\sqrt{\boldsymbol{v}}} \mathcal{Q} \frac{1}{\sqrt{\boldsymbol{v}}} = M_{vv} - M_{vu} \frac{1}{1 + M_{uu}} M_{uv} \tag{26}$$

The $\boldsymbol{S}$ matrix is defined via

$$\begin{pmatrix} \tilde{B}_L \\ \tilde{B}_R \end{pmatrix} = \boldsymbol{S} \begin{pmatrix} \tilde{A}_L \\ \tilde{A}_R \end{pmatrix} \tag{27}$$

and can be written in block form as

$$\boldsymbol{S_{n,m}} = \begin{pmatrix} \boldsymbol{S}_R & \boldsymbol{S}_T \\ \boldsymbol{S}_T & \boldsymbol{S}_R \end{pmatrix}$$

A straightforward elimination gives

$$\boldsymbol{S} = \begin{pmatrix} -\boldsymbol{M}_{--}^{-1}\boldsymbol{M}_{-+} & \boldsymbol{M}_{--}^{-1} \\ \boldsymbol{M}_{++} - \boldsymbol{M}_{-+}\boldsymbol{M}_{--}^{-1}\boldsymbol{M}_{+-} & \boldsymbol{M}_{+-}\boldsymbol{M}_{--}^{-1} \end{pmatrix} = \begin{pmatrix} (1 + i\mathcal{M})^{-1} - 1 & (1 + i\mathcal{M})^{-1} \\ (1 + i\mathcal{M})^{-1} & (1 + i\mathcal{M})^{-1} - 1 \end{pmatrix} \tag{28}$$

Now we can write expressions for $\boldsymbol{S}_R$ and for $\boldsymbol{S}_T$ using the $\mathcal{M}$ matrix.

$$\begin{aligned} \boldsymbol{S}_T &= \frac{1}{1 + i\mathcal{M}} = 1 - i\mathcal{M} - \mathcal{M}^2 + i\mathcal{M}^3 + ... && (29) \\ \boldsymbol{S}_R &= \boldsymbol{S}_T - \boldsymbol{1} && (30) \end{aligned}$$