Ex258: Scattering of particle by a delta function in waveguide

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The problem:

Find the S matrix for a particle in a waveguide of width W with a delta scatterer $V(x, y) = c\ \delta(x)\delta(y - y_0)$. Explain the effect of the closed channels on the total reflection, and on the ability of a delta barrier to scatter in 1D. Estimate the maximum reflection if closed channels could be ignored.

Tip: Given a column vector a, we can form a matrix $A = aa^t$, such that trace$[A] = aa^t$, and we have the identity:

$$\frac{1}{1 - aa^t} = 1 + (aa^t) + (aa^t)^2 + (aa^t)^3 ... = 1 + a[1 + (a^t a) + (a^t a)^2 + ...]a^t = 1 + \frac{1}{1 - a^t a} aa^t \tag{1}$$

hence

$$\frac{1}{1 - A} = 1 + \frac{1}{1 - \text{trace}[A]} A \tag{2}$$

The solution:

The Hamiltonian is:

$$\mathcal{H} = \frac{p_x^2}{2m} + \frac{p_y^2}{2\mathsf{m}} + V_{\text{walls}}(y) + V(x, y) \tag{3}$$

Where:

$V_{\text{walls}}(y) =$ the potential formed by the boundary conditions of the waveguide.

$V(x, y) =$ the potential formed by the delta scatterer.

Outside the scattering region, the total energy can be expressed by:

$$E_{\text{tot}} = \frac{k^2}{2m} + \frac{1}{2m}\left(\frac{\pi}{W} n_y\right)^2 = \epsilon_k + E_{n_y} \tag{4}$$

In order to define a channel we need to know in which side is the wave (n_0) and what is its mode (n_y).

$$n = (n_0, n_y)$$

We define for open channel

$$k_n = \sqrt{2\mathsf{m}(E_{\text{tot}} - E_n)} \tag{5}$$

and for closed channels:

$$\alpha_n = \sqrt{-2\mathsf{m}(E_{\text{tot}} - E_n)} \tag{6}$$

and use the notations

$$v_{\boldsymbol{n}} = k_{\boldsymbol{n}}/\mathsf{m} \tag{7}$$

$$u_{\boldsymbol{n}} = \alpha_{\boldsymbol{n}}/\mathsf{m} \tag{8}$$

The wavefunction can be written as

$$\Psi(r, n_0, y) \quad = \quad \sum_{n_y} R_{n_0, n_y}(r)\ \chi^{n_y}(y) \tag{9}$$

where

$$\chi^{n_y}(y) = \sqrt{\frac{2}{W}} \sin(k_n y) \tag{10}$$

The matching conditions are

$$\Psi(0,\text{right},y) = \Psi(0,\text{left},y) \tag{11}$$

$$\frac{1}{2\mathsf{m}}\left[\Psi'(0,\text{right},y)+\Psi'(0,\text{left},y)\right] = \hat{Q}\Psi(0,y) \tag{12}$$

Where

$$\hat{Q} = c\delta(y-y_0) \tag{13}$$

The matrix elements of $\hat{Q}$ are

$$Q_{\text{nm}} \;=\; c\int \chi_n\delta(y-y_0)\chi_m \text{dy} \;=\; u\;\chi^n(y_0)\chi^m(y_0) \tag{14}$$

We define the 4×4 block matrix

$$M_{nm} \;=\; \frac{1}{\sqrt{|v_n|}}Q_{nm}\frac{1}{\sqrt{|v_m|}} \;\equiv\; a_n a_m \;\equiv\; \begin{pmatrix} M_{vv} & M_{vu} \\ M_{uv} & M_{uu}\end{pmatrix} \tag{15}$$

where $|v_n|$ is v_n or u_n depending on whether n is open or closed, and the column vector a_n is defined as follows:

$$a_n = \sqrt{c}\frac{1}{\sqrt{|v_n|}}\chi^n(y_0) \tag{16}$$

Following the same procedure as in **Ex.259** we define:

$$\mathcal{M} = M_{vv} - M_{vu}\frac{1}{(\mathbf{1}+M_{uu})}M_{uv} \tag{17}$$

Using the identity that was mentioned as a "tip" we find

$$\mathcal{M} = \frac{1}{1+\text{trace}[M_{uu}]}M_{vv} \tag{18}$$

we see that the closed channels 'renormalize' the matching matrix, and hence make it energy dependent.

Going on with the procedure of **Ex.259** we find the following expressions for the reflection and the transmission blocks of the scattering matrix:

$$\boldsymbol{S}_T \;=\; \frac{1}{1+i\mathcal{M}} \tag{19}$$

$$\boldsymbol{S}_R \;=\; S_T - 1 \tag{20}$$

Using a second time the identity that was mentioned as a "tip" we find

$$\boldsymbol{S}_R = \frac{iM_{vv}}{1+i\;\text{trace}[M_{vv}]+\text{trace}[M_{uu}]} \tag{21}$$

The major observations are: The number of open channels is $N \sim kW$. The first order result for the reflection amplitude is $|r| \sim c/(Wv)$. The total reflection probability within this approximation is $p \sim N|r|^2$ leading to $p \sim (\mathsf{m}c)^2/N$. Disregarding the closed channels p is bounded by $1/N$ because in the denominator of the reflection amplitude we have $(1+\text{trace}[M_{vv}]) \sim (1+\mathsf{m}c)$. Note that the $1/N$ bound on the total reflection can be deduced in advance from unitarity if we assume isotropic scattering (to all channels) with real amplitudes. As a function of energy whenever a channel is opened we have zero reflection. If we take all the closed channels into account (without cutoff) the reflection goes to zero irrespective of energy.