Ex2554: General S-Matrix of a 3 channel junction

Submitted by: Tal Gur, Amitai Degani

Problem description:

Given a 3-channel junction with a potential energy parameter u, find the most general form of the S-Matrix which satisfies the conditions:

(1) Continuity at $x = 0$: $\Psi_1(0) = \Psi_2(0) = \Psi_3(0)$

(2) Non-continuity of the derivative at $x = 0$: $\Sigma_{i=1}^{3} \frac{\partial \Psi_i}{\partial x}|_{x=0} = 3mu\Psi(0)$

Where x is defined to be the distance from the junction (essentially $|x|$)

Solution:

We'll begin by examining a general N channel junction, with a single incident wave from one channel.

Incident and returning wave: $\Psi_i = e^{-ikx} - re^{ikx}$

Transmitted wave: $\Psi_t = te^{ikx}$

Where $k = \sqrt{2mE} = mv_E$

We have one channel which consists of the Incident and Return wave, and N-1 channels with one Transmitted wave each. The conditions can be rewritten as follows:

(1) $\forall i, j : \Psi_i(0) = \Psi_j(0)$

(2) $\Sigma_{i=1}^{N} \frac{\partial \Psi_i}{\partial x}|_{x=0} = Nmu\Psi(0)$

We shall assume the junction is completely symmetrical in it's transfer, meaning all the transmitted parameters t are equal to one another.

From condition (1):

$\Psi_i(0) = \Psi_t(0)$

$1 + r = t \quad \Rightarrow \quad r = t - 1$

From condition (2):

$\frac{\partial \Psi_i}{\partial x}|_{x=0} + (N-1)\frac{\partial \Psi_t}{\partial x}|_{x=0} = Nmu\Psi_t(0)$

$-ik + ikr + (N-1)ikt = Nmut$

$-ik(1 - r - (N-1)t) = Nmut$

$1 - t + 1 - (N-1)t = iNtu/v_E$

$2 - Nt = iNtu/v_E$

$2 = Nt(1 + iu/v_E)$

$t = \frac{2}{N}\frac{1}{1+iu/v_E}$

$r = t - 1 = \frac{2}{N}\frac{1}{1+iu/v_E} - 1$

For the 3 channel junction, $N = 3$:

$t = \frac{2}{3}\frac{1}{1+iu/v_E}$

$r = t - 1 = \frac{2}{3}\frac{1}{1+iu/v_E} - 1$

And so we can write the S-matrix as:

$$\mathcal{S} = -\begin{pmatrix} r & t & t \\ t & r & t \\ t & t & r \end{pmatrix}$$

And the general element is $S_{ab} = \delta_{ab} - \frac{2}{3}\frac{1}{1+iu/v_E}$.

We can see that in the extreme case of $u = 0$, S becomes the normal uninterrupted junction matrix

$$\mathcal{S} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

And in the case of $u \to \infty \Rightarrow r \to 1$ and $t \to 0$ and we get full reflection.

If we define as follows:

$\gamma = arg(t) = -arctan(\frac{u}{v_E}) \quad mod\ \pi$

$g = |\frac{2}{3}\frac{1}{1+iu/v_E}|^2 = \frac{4}{9}\frac{1}{1+(u/v_E)^2}$

Then we can write S as:

$$\mathcal{S} = -e^{i\gamma}\begin{pmatrix} -i\sqrt{1-2g}e^{i\alpha} & \sqrt{g}e^{-i\beta} & \sqrt{g}e^{-i\beta} \\ \sqrt{g}e^{i\beta} & -i\sqrt{1-2g}e^{i\alpha} & \sqrt{g}e^{-i\beta} \\ \sqrt{g}e^{i\beta} & \sqrt{g}e^{i\beta} & -i\sqrt{1-2g}e^{i\alpha} \end{pmatrix}$$