Ex264: Junction scattering matrix

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The problem:

Three strings are connected at one point, a signal is sent from one string and scatters to the other two strings and to itself.

(1) Find the the scattering matrix (r- return amplitude, t- transfer amplitude).
(2) Find the value r as function of the phase between the return amplitude and the transfer amplitude.
(3) What is the minimal value of r ?
(4) Generalize the solution for the case of more than 3 strings.
(5) What is the value of t,r where the number of strings aspire to infinite ?

The solution:

(1) based on the question we are talking about a 3x3 matrix

$$S = \begin{pmatrix} r & t & t \\ t & r & t \\ t & t & r \end{pmatrix}$$

(2) thanks to the calibration freedom, we will set the amplitudes as such:

$$r = r_0 e^{i\alpha}$$

$$t = t_0 e^{i\beta}$$

(3)

$$S^\dagger = (S^T)^* = \begin{pmatrix} r^* & t^* & t^* \\ t^* & r^* & t^* \\ t^* & t^* & r^* \end{pmatrix}$$

because the scattering matrix is known to be unitary :

$$SS^\dagger = \begin{pmatrix} r & t & t \\ t & r & t \\ t & t & r \end{pmatrix} \begin{pmatrix} r^* & t^* & t^* \\ t^* & r^* & t^* \\ t^* & t^* & r^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} rr^* + 2tt^* = 1 \\ rt^* + r^*t + tt^* = 0 \end{cases}$$

$$\Rightarrow \begin{cases} r_0^2 + 2t_0^2 = 1 \\ r_0 t_0 (e^{i(\alpha-\beta)} + e^{-i(\alpha-\beta)}) + t_0^2 = 0 \end{cases}$$

$$t_0(2r_0\cos(\alpha-\beta) + t_0) = 0 \qquad \Rightarrow \boldsymbol{t_0 = -2r_0\cos(\alpha-\beta)}$$

$$r_0^2(1 + 8\cos^2(\alpha-\beta)) = 1 \qquad \Rightarrow r_0 = \sqrt{\frac{1}{1+8\cos^2(\alpha-\beta)}} \qquad \Rightarrow [for(\alpha-\beta)=0] \Rightarrow \boldsymbol{r_{0min} = |\pm\frac{1}{3}| = \frac{1}{3}}$$

(4) we shall asume N strings our new scattering matrix will be an NxN matrix the equetion that we will get is:

$$\Rightarrow \begin{cases} rr^* + (N-1)tt^* = 1 \\ rt^* + r^*t + (N-2)tt^* = 0 \end{cases}$$

(5) we shall take N to be infinity and solve the equations:

$$2r_0 t_0 \cos(\alpha-\beta) + (N-2)t_0^2 = 0 \qquad \Rightarrow t_0 = \frac{2r_0\cos(\alpha-\beta)}{(2-N)}$$

$$r_0^2 + (N-1)t_0^2 = 1 \qquad \Rightarrow r_0^2 = 1 + \frac{(1-N)4\cos^2(\alpha-\beta)r_0^2}{(2-N)^2} \qquad \Rightarrow r_0 = \sqrt{\frac{(2-N)^2}{(2-N)^2+(N-1)4\cos^2(\alpha-\beta)}}$$

$$\Rightarrow [N\to\infty] \Rightarrow \begin{cases} r_0 \to & 1 \\ t_0 \to & 0 \end{cases}$$