E558: Particle in a moving-wall box

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The problem:

A particle is trapped inside a $1D$ time-dependent-wall box $0 < x < L(t)$. Show that using *dilation* one can transform and obtain a constant-wall box Hamiltonian.

Tip: The dilation generator is defined as $G = \frac{1}{2}(xp + px)$.

1. Write the resulting Hamiltonian
2. Identify vector potential $A(x)$
3. Identify scalar potential $V(x)$
4. What gauge transformation is required in order to vanish the vector potential
5. What is the expression of the resulting scalar potential $\tilde{V}(x)$ after the gauge transformation

The solution:

1 Since the Hamiltonian in the old frame (constant-wall box $0 < x < 1$) form $H = h(x, p, V, A)$ can be rewritten in the new frame (time-dependent-wall box $0 < x < L(t)$) as

$$\tilde{H} = h(SxS^{-1}, SpS^{-1}, V, A) + \imath\frac{\partial S}{\partial t}S^{-1} \quad \text{(see D.C. Lecture Notes 17.1, p.73)}$$

where S is a dilation operator (see *Tip* and *Appendix*) defined as

$$S = \mathrm{e}^{\imath \ln L(t) G}$$

one can immediately obtain

$$\tilde{H} = \frac{p^2}{2mL^2} - \frac{1}{2}\frac{\dot{L}}{L}(xp + px)$$

where the following identities (1) are used

$$SxS^{-1} = Lx, \qquad SpS^{-1} = \frac{p}{L}, \qquad \imath\frac{\partial S}{\partial t}S^{-1} = -\frac{\dot{L}}{L}G = -\frac{1}{2}\frac{\dot{L}}{L}(xp + px) \tag{1}$$

2,3 Rewriting the obtained Hamiltonian in the following way

$$\tilde{H} = \frac{1}{2m}\left(\frac{p^2}{L^2} - m\frac{\dot{L}}{L}xp - m\frac{\dot{L}}{L}px\right) = \frac{1}{2mL^2}(p - mL\dot{L}x)^2 - \frac{1}{2}m\dot{L}^2x^2$$

one can identify vector potential, which is

$$A(x) = mL\dot{L}x$$

and scalar potential, which is

$$V(x) = -\frac{1}{2}m\dot{L}^2x^2$$

4 Since the Hamiltonian is invariant with

$$\tilde{A}(x) = A(x) + \nabla\Lambda(x)$$

in order to vanish the vector potential one has to consider the case $\tilde{A}(x) = 0$, therefore

$$\nabla\Lambda(x) = -A(x) \quad \Rightarrow \quad \Lambda(x) = -\int A(x)dx = -\int mL\dot{L}xdx$$

leading to

$$\Lambda(x) = -\frac{1}{2}mL\dot{L}x^2$$

5 After applying gauge transformation the resulting scalar potential is

$$\tilde{V}(x) = V(x) - \frac{\partial\Lambda}{\partial t} = \frac{1}{2}mL\ddot{L}x^2$$

Appendix

By definition

$$S_\lambda\Psi(x) = \Psi(\mathrm{e}^\lambda x)$$

which is identical transformation for $\lambda = 0$. By expanding $\Psi(\mathrm{e}^\lambda x)$ in series, one can obtain

$$\Psi(\mathrm{e}^\lambda x) = \Psi(x) + x\frac{\partial\Psi}{\partial x}\lambda + \left(x\frac{\partial\Psi}{\partial x} + x^2\frac{\partial^2\Psi}{\partial x^2}\right)\frac{\lambda^2}{2} + \left(x\frac{\partial\Psi}{\partial x} + 3x^2\frac{\partial^2\Psi}{\partial x^2} + x^3\frac{\partial^3\Psi}{\partial x^3}\right)\frac{\lambda^3}{6} + \ldots$$

or introducing momentum operator $p = -\imath\frac{\partial}{\partial x}$ and using commutator identity $[p, x] = -\imath$, one obtains

$$x\frac{\partial}{\partial x} = \imath xp - \frac{1}{2} + \frac{1}{2} = \imath G - \frac{1}{2},$$

$$x\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial x^2} = x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right) = \left(\imath G - \frac{1}{2}\right)^2,$$

and so on ....

Then one obtains

$$\Psi(\mathrm{e}^\lambda x) = \left[1 + \left(\imath G - \frac{1}{2}\right)\lambda + \left(\imath G - \frac{1}{2}\right)^2\frac{\lambda^2}{2} + \ldots\right]\Psi(x)$$

The latter expression inside the square brackets is the dilation operator

$$S_\lambda = \mathrm{e}^{\imath\lambda G}\mathrm{e}^{-\frac{\lambda}{2}}$$

Demanding $\mathrm{e}^\lambda = L(t) \Rightarrow \lambda = \ln L(t)$ one obtains

$$S_L = \frac{1}{\sqrt{L(t)}}\mathrm{e}^{\imath\ln L(t)G}$$

Then

$$\Psi(L(t)x) = \frac{1}{\sqrt{L(t)}}\mathrm{e}^{\imath\ln L(t)G}\Psi(x)$$

or

$$\mathrm{e}^{\imath \ln L(t) G} \Psi(x) = \sqrt{L(t)} \Psi(L(t) x)$$

The inverse operator

$$S_L^{-1} = S_{\frac{1}{L}} = \sqrt{L(t)} \mathrm{e}^{-\imath \ln L(t) G}$$

and

$$\mathrm{e}^{-\imath \ln L(t) G} \Psi(x) = \frac{1}{\sqrt{L(t)}} \Psi\left(\frac{x}{L}\right)$$

Here, in order to keep the norm of $\Psi(x)$ unchanged, the dilation operator is chosen as

$$S = \mathrm{e}^{\imath \ln L(t) G}$$

which is a unitary operator i.e. $S^{\dagger} = S^{-1}$.