E556: A particle in a moving box - elegant way of solution

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The problem:

A particle is placed in a moving box, with a velocity of (u). $(ut < x < ut + a)$.
The particle is held at the energy level (n).

(1) write the wave function of the particle in the laboratory system.
(2) explain the analogy to the classical solution (distribution in phase space).

The solution:
(1) The Hamiltonian in the Laboratory system is:

$$H = \frac{p^2}{2m} + V(x - ut) \tag{1}$$

transformation from the laboratory system to the moving box system:

$$T = ee^{imu\hat{x}}e^{-iut\hat{p}} \tag{2}$$

note that operating the bost and the displacement in a diffrent order derives phase!

in the S formalism:

$$S = T^{-1} = e^{iut\hat{p}}e^{-imu\hat{x}} \tag{3}$$

for

$$(AB)^{-1} = B^{-1}A^{-1}$$

the hemiltonian in the moving box is:

$$\tilde{H} = SHS^{-1} + i\frac{\partial S}{\partial t}S^{-1} \tag{4}$$

$$SHS^{-1} = H(S\hat{x}S^{-1}, S\hat{p}S^{-1}) = H(\hat{x} + ut, \hat{p} + mu) \tag{5}$$

$$i\frac{\partial S}{\partial t}S^{-1} = -u\hat{p} \tag{6}$$

$$\tilde{H} = \frac{(\hat{p} + mu)^2}{2m} + V(x) - up = \frac{\hat{p}^2}{2m} + \frac{1}{2}mu + V(x) \tag{7}$$

We will operate the hemiltonian on the wave funcion in the moving box system.
The wave funcion in the moving box system is knon as:

$$\tilde{\varphi}(x) = \sqrt{\frac{2}{a}}\sin(k_n x) \tag{8}$$

$$\tilde{H}\tilde{\psi}(x) = E_n\tilde{\psi}(x) \tag{9}$$

$$(\frac{\hat{p}^2}{2m} + \frac{1}{2}mu^2)\sqrt{\frac{2}{a}}\sin(k_n x) = (\frac{-1}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}mu^2)\sqrt{\frac{2}{a}}\sin(k_n x) = \tag{10}$$

$$(\frac{k^2}{2m} + \frac{1}{2}mu^2)\sqrt{\frac{2}{a}}\sin(k_n x) \tag{11}$$

$$E_n = \frac{k^2}{2m} + \frac{1}{2}mu^2 \tag{12}$$

time dependent wave function:

$$\tilde{\psi}(x) \ = \ \tilde{\varphi}(x)e^{-iE_n t} = \sqrt{\frac{2}{a}}e^{-iE_n t}\sin(k_n x) \tag{13}$$

Oparating the inverse transphorm will take us back to the laboratory system

$$S^{-1}\tilde{\psi}(x) \ = \ \psi(x) \tag{14}$$

$$S^{-1}\tilde{\psi}(x) = e^{-iE_n t}\langle x|e^{-iE_n t}e^{imu\hat{x}}e^{-iut\hat{p}}|n\rangle = e^{-iE_n t}\sum_{\acute{x}}\langle x|e^{ima\acute{x}}|\acute{x}\rangle\langle\acute{x}|e^{-iut\hat{p}}|n\rangle = \tag{15}$$

$$e^{-iE_n t}\sum_{\acute{x}}e^{ima\acute{x}}\langle x|\acute{x}\rangle\langle\acute{x} - ut|n\rangle = e^{-iE_n t}\sum_{\acute{x}}e^{ima\acute{x}}\delta_{x,\acute{x}}\langle\acute{x} - ut|n\rangle = e^{-iE_n t}e^{imax}\sin(k(x - ut))$$

$$\psi(x) = \sqrt{\frac{2}{a}}e^{-i(\frac{k^2}{2m}+\frac{1}{2}mu^2)t}e^{imux}\sin(k(x - ut)) \tag{16}$$

(2) classical point of view means we can see a particle going back and forth in a box. it performs an elastic collision with we walls of the box. at the laboratory system we can write:

$$p_1 = mu + \bar{p} = mu + \sqrt{2mE} \tag{17}$$

$$p_2 = mu - \bar{p} = mu - \sqrt{2mE}$$

the wave function is a superposition of the two free waves:

$$\tilde{\varphi}(x) = Ae^{i(mu-\sqrt{2mE})x} + Be^{i(mu+\sqrt{2mE})x} \tag{18}$$