E204: A particle is in a box that moves at speed u.

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The problem:

A particle is in a box that moves at speed u. i.e., $ut < x < ut + a$. Since the problem is dependent on time there are no stationary states in the laboratory system. It is possible to find a solution to Schrodinger's Equation which is dependent on time by separation of variables and has the form:$\Psi(x,t) = \psi(x - ut)f(t)$.
1. Find these solutions (before you solve, try to guess the answer ahead).
2. Explain the analogy to the classic stationary solution (distribution in the space of the phases).

Solution
(1) A particle is found in a box that drifts in constant velocity u. i.e. $ut < x < ut + a$. since the problem is dependent on time there is no stationary states in the lab system. In order to find a solution to Schrodinger equation dependent on time we can use parting variable technique:$\Psi(x,t) = \psi(x - ut)f(t)$.

The potential $V(x,t)$ in the problem is defined as: $V(x,t) = 0$ at $ut < x < ut + a$ (a is the length of the box), otherwise (out of the box) the potential is infinity.

Schrodinger equation depends on time:

$$i\frac{\partial\Psi(x,t)}{\partial t} = (-\frac{\partial^2}{2m\partial x^2} + V(x - ut))\Psi(x,t)$$

Note: it is customary to refer to $\hbar$ as 1.

The boundary limit:
1.$\Psi(ut,t) = 0$
2.$\Psi(x + ut, t) = 0$

In order to get rid of the time dependent boundary conditions we define new variables (r,τ) instead of (x,t):

$$r = x - ut$$

$$t = \tau$$

And we define:

$$\tilde{\Psi}(r,\tau) = \Psi(x,t)$$

we can write the equation for $\tilde{\Psi}(r,\tau)$ using:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - u\frac{\partial}{\partial r}$$

This leads to the equation:

$$\frac{\partial\tilde{\Psi}(r,\tau)}{\partial t} = i\frac{1}{2m}\frac{\partial^2\tilde{\Psi}(r,\tau)}{\partial^2 r} + u\frac{\partial\tilde{\Psi}(r,\tau)}{\partial r} - iV(r)\tilde{\Psi}(r,\tau)$$

Considering the boundary conditions:

$$\tilde{\Psi}(0,\tau) = \tilde{\Psi}(a,\tau) = 0$$

This equation is of the standard Schrodinger type. We solve it by looking for solutions of the form:

$$\tilde{\Psi}(r,\tau) = f(\tau)\psi(r)$$

with:

$$f(\tau) = \exp(-iE\tau)$$

Then we get for $\psi(r)$ the eqnation:

$$\frac{\partial^2\psi(r)}{\partial^2 r} - (2miu)\frac{\partial\psi(r)}{\partial r} + (2mE)\psi(r) = 0$$

Note: $V(r) = 0$ in the box.

The general solution for $\psi(r)$:

$$\psi(r) = A * \exp(ik_+ r) + B * \exp(ik_- r)$$

where:

$$k_\pm = \pm k_n + mu = \pm\sqrt{2mE_n} + mu \qquad n = 1, 2, 3......$$

$$E_n = \frac{\pi^2 n^2}{2ma^2} - \frac{mu^2}{2} \qquad n = 1, 2, 3......$$

Considering the boundary conditions:

$$\psi(0) = \psi(a) = 0$$

The solution to the above equation will include only the *sin* part:

$$\psi^n(r) = \exp{(imur)}\sin(k_n r) \qquad n = 1, 2, 3.......$$

$$\tilde{\Psi}^n(r,\tau) = \psi^n(r)f(\tau) = \sqrt{\frac{2}{a}}\exp(-iE_n\tau)\exp{(imur)}\sin(k_n r) \qquad n = 1, 2, 3.......$$

$$\Psi^n(x,t) = \psi^n(x-ut)f(t) = \sqrt{\frac{2}{a}}\exp(-iE_n t)\exp{(imu(x-ut))}\sin(k_n(x-ut)) n = 1, 2, 3.......$$

$$\Psi^n(x,t) = \sqrt{\frac{2}{a}}\exp(-i(\frac{\pi^2 n^2}{2ma^2} + \frac{mu^2}{2})t)\exp{(imux)}\sin(k_n(x-ut)) \qquad n = 1, 2, 3.......$$

The general solution is:

$$\Psi(x,t) = \sum C_n \Psi^n(x,t)$$

where C_n are arbitrary.

Checking the Solution:

The classic argument is that $V(x,t) = 0$ when $0 < x < a$, otherwise $V(x,t)$ equales infinity. Another fact considering the classic argument is that the velocity u equals 0 at all times. So, by inserting $u = 0$ in the $\Psi^n(x,t)$, that was formed above, we will get the classic term:

$$\Psi^n(x,t) = \sqrt{\frac{2}{a}} exp(-iE_n t) \cdot \exp(im * 0 * (x - 0 * t)) \cdot \sin(k_n(x - 0 * t))$$

$$k_n = \sqrt{2mE_n} \qquad n = 1, 2, 3.....$$

$$E_n = \frac{\pi^2 n^2}{2ma^2} - \frac{m * 0^2}{2} = \frac{\pi^2 n^2}{2ma^2} \qquad n = 1, 2.....$$

$$\Psi^n(x,t) = \sqrt{\frac{2}{a}} \exp(-iE_n t) \cdot \sin(k_n \cdot x) \qquad n = 1, 2...$$

(2) In the first part we can see the classic argument derivative from the general solution. The main difference between the two arguments is in their phases space that is dependent on k(momentum). In the classic case:

$$\Psi^n(x,t) = A * \exp(ik_n r) + B * \exp(-ik_n r)$$

Where:

$$k_n = \sqrt{2mE_n} \qquad n = 1, 2, 3.....$$

In this problem:

$$\Psi^n(x,t) = A * \exp(i(k_n + mu)r) + B * \exp(-(ik_n - mu)r) = A * \exp(ik_+ r) + B * \exp(i(ik_- r)$$

Where $k_\pm$ represent the momentum of the particle in both direction (positive and negative). The velocity of the particle in general can bey written as the velocity of the particle in the classic argument plus the velocity of the box u to give: $v_\pm = \pm\sqrt{2E/m} + u$.