E209: A particle on a Klein bottle

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The problem:

1. Write what are the boundary conditions (B.C.) that characterize a Klein bottle.
2. Find the eigenstates and the energy levels of a particle confined to move on a Klein bottle.

The solution:

(1) Klein bottle can be represented by a rectangle of width L_x and length L_y, with periodic B.C. in the y direction and Mobius like B.C. in the x direction.
Periodic B.C. in the y direction are:

$$a)\quad \psi(x,\frac{L_y}{2}) = \psi(x,-\frac{L_y}{2})$$

$$b)\quad \partial_y\psi(x,\frac{L_y}{2}) = \partial_y\psi(x,-\frac{L_y}{2})$$

Mobius like B.C. in the x direction are:

$$a)\quad \psi(-\frac{L_x}{2},y) = \psi(\frac{L_x}{2},-y)$$

$$b)\quad \partial_x\psi(-\frac{L_x}{2},y) = \partial_x\psi(\frac{L_x}{2},-y)$$

(2) The eigenvalues equation is separable, therefore its general solution is:

$$\psi\left(x,y\right) = A_1\cos\left(k_y y\right)\sin\left(k_x x+\varphi\right) + A_2\sin\left(k_y y\right)\sin\left(k_x x+\varphi\right)$$

where φ is free. This means that there is a degeneracy due to symmetry under translation in the x direction. Without loss of generality we take $\varphi = 0$.

(3) The periodic B.C. on the y direction imply:

$$k_y = \frac{2\pi}{L_y}n_y \quad (n_y \;=\; integer).$$

(4)In case of periodic boundary conditions there is degeneracy due to the translational combined with mirror symmetry in the y direction.The Klein boundary conditions destroys the translational symmetry but leave the mirror symmetry. Therefore the degeneracy is removed and the eigenstates are divided into even (A2=0) and odd (A1=0).

(5) In order to determine k_x we have to consider separately the even states and the odd states.
It can be easily shown that if we had periodic B.C. in the x direction then:

$$k_x = \frac{2\pi}{L_x}n_x \quad (n_x \;=\; integer),$$

With Klein B.C., however, this is still correct only for the even states, but for the odd states we get:

$$k_{x_{odd}} = \frac{2n_x\pi + \pi}{L_x} \qquad (n_x \ = \ integer).$$

In other words, only the odd states are affected when we switch from a conventional cylinder (periodic B.C.) to a Klein bottle.

(6) Finally we can calculate the energy:

$$E_{even/odd} = \frac{\hbar^2}{2m}\left(k^2_{x_{even/odd}} + k^2_y\right)$$

Unlike the case with Mobius here for each n_y we have two eigenstates: one is even, and one is odd. In case of Mobius for each n_y we had either even or odd eigenstate.