Ex: E2080: A particle on a Mobius strip

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The problem:

Find the energies and eigenstates of a particle confined to move on a Mobius strip of size $L_x \times L_y$, with the boundary conditions:

$a)$ $\psi(x, 0) = 0$

$b)$ $\psi(x, L_y) = 0$

$c)$ $\psi(x + L_x, y) = \psi(x, -y)$

The solution:

In the bulk, the Hamiltonian is of a free particle.

$$\mathcal{H} \;=\; \frac{P_x^2}{2m} \;+\; \frac{P_y^2}{2m}$$

The general solution is:

$$\psi = A\mathrm{e}^{-i(k_x x + k_y y)} + B\mathrm{e}^{i(k_x x + k_y y)}$$

From conditions (a) and (b) we get $A\mathrm{e}^{-ik_x x} Sin(k_y L_y) = 0$, which give us the values for k_y:

$$k_y = \frac{\pi l}{L_y} \;; \quad l = 1, 2, ..$$

From condition (c) we get $\mathrm{e}^{-ik_x L_x} = -1$, which yields the values of k_x of the form:

$$k_x = \frac{\pi(2n-1)}{L_x} \;; \quad n = 1, 2, ..$$

The wave function takes the form $\psi = \sum_{l,n=1}^{\infty} D\mathrm{e}^{-ik_x x} Sin(k_y y)$. Normalizing the wave function will give us D:

$$|\psi|^2 = \int_0^{L_x}\int_0^{L_y} |D|^2 Sin^2(k_y y) dx dy = |D|^2 \frac{L_x L_y}{2} = 1$$

so the eigenstates are:

$$\psi_{l,n}(x, y) = \sqrt{\frac{2}{L_x L_y}}\mathrm{e}^{-i\frac{\pi(2n-1)}{L_x}x} Sin(\frac{\pi l}{L_y} y)$$

with eigenvalues:

$$E_{n,l} = \frac{P_x^2}{2m} \;+\; \frac{P_y^2}{2m} = \frac{\pi^2(2n-1)^2}{2mL_x^2} \;+\; \frac{\pi^2 l^2}{2mL_y^2}$$