E144: Quantum Hotel

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The problem:

In a quantum hotel there are four identical rooms connected in a row(column). It's well know that it someone is present in one of the rooms at t=0, there is a chance that he will be present in one of the adjacent rooms in a differnt time (t=0). The walls separating the rooms are identical except the wall that separates between room 2 and 3. (The purpose of the drill is to allow exercizing the use of symmetry that exist in the problem, in order to simplify the search for the independent states).

(1)Write down the operator $\widehat{M}$, that discribes the measuring of the room number in a matrix.
(2)Write down Hameltonian $\widehat{H}$ in the base in which $\widehat{M}$ is diagonal. Add expressions for necessery constants and express their maening. It's well visible that the system is symmetric under a specific exchange of numbering of rooms.
(3)Show this exchange in a matrix of operator $\widehat{F}$ that is exchangeable with Hameltonian.
(4)Express $\widehat{H}$ in a base where $\widehat{F}$ is diagonal and find the eigenvalue of $\widehat{H}$.

The solution:

1)We describe the state of being in room number i with the state vector $|i\rangle$. The operator that returns the room number will be defined as $\widehat{M}\,|i\rangle$=i$|i\rangle$ and it's matrix representation would be:

$$\widehat{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

2)The physical meaning of elements H_{ij} in the Hamiltonian is the Probability Amplitude per second of the particle to move from site i to site j.Accordingly, if the Hamiltonian is real we have:

$$\widehat{H} = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & b & 0 \\ 0 & b & 0 & a \\ 0 & 0 & a & 0 \end{pmatrix}$$

3)the system remains unchanged with the room transformation $1 \longleftrightarrow 4, 2 \longleftrightarrow 3$. because of the reflaction symmetry the Hameltonian is unchanged in the reflaction basis. The matrix that reflects with kind of transformation is:

$$\widehat{F} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

And it comutes with $\widehat{H}$ because

$$\widehat{H}\widehat{F} = \widehat{F}\widehat{H} = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}$$

4)We express the eigenvectors of the diagonalized $\widehat{F}$ matrix in tje position basis:

$$|S_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |4\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$|S_2\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

$$|A_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |4\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

$$|A_2\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

And the eigenvalues are:

$$\widehat{F}|S_1\rangle = 1|S_1\rangle, \qquad \widehat{F}|S_2\rangle = 1|S_2\rangle$$
$$\widehat{F}|A_1\rangle = -1|A_1\rangle, \qquad \widehat{F}|A_2\rangle = -1|A_2\rangle$$

The transformation matrix $\widehat{T}$ from the old basis to the new is:

$$\widehat{T} = \begin{pmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\0 & 1 & 0 & -1\\1 & 0 & -1 & 0\end{pmatrix}$$

According to $\widehat{T}^{-1}$=$\widehat{T}^{\dagger}$ we obtain $\widehat{H}$ in the basis of $\widehat{F}$ is:

$$\widetilde{H} = T^{-1}HT = \begin{pmatrix}0 & a & 0 & 0\\a & b & 0 & 0\\0 & 0 & 0 & a\\0 & 0 & a & -b\end{pmatrix}$$

Due to the above variable separation we got a block diagonal matrix and can diagonalize each block separetly The eignvalues of $\widetilde{H}$ are:

$$E_n = \frac{\pm b \pm \sqrt{4a^2 + b}}{2}$$