E141: 3 sites system with leaking

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The problem:

A particle can be found in one of three sites $|1,2,3\rangle$ (all sites have the same binding energy).
The jump amplitude per time unit from site 1 to 2 is Δ_0.
The jump amplitude per time unit from site 2 to 3 is c.
At $t = 0$ the particle is placed in site number 2.
The goal of the question is to find the probability P(t) for the particle to be at site number $|3\rangle$.

(1) Write down the system Hamiltonian.

Assume that c=0.
(2) Write down the expression for the survival amplitude $\psi_2(t)$ of being in site 2.
(3) Write down the expression for $\frac{d\psi_3(t)}{dt}$, and find the first order solution for $\psi_3(t)$.

Assume that $c \neq 0$. The goal is to find the exact solution for P(t).
(4) Find the eigenvalues and eigenstates of the system.
(5) Explain why one of the eigenstates is irrelevant to the solution.
(6) What is the oscillation frequency of the particle?
(7) Find the exact solution for P(t).

The solution:

(1) Defining

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

The Hamiltonian is:

$$\hat{\mathcal{H}} = \begin{pmatrix} 0 & \Delta_0 & 0 \\ \Delta_0 & 0 & c \\ 0 & c & 0 \end{pmatrix}.$$

(2) Assuming c=0, the Hamiltonian is now:

$$\hat{\mathcal{H}} = \begin{pmatrix} 0 & \Delta_0 & 0 \\ \Delta_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenstates are:

$$|S\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad |A\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and the eigenvalues are given by:

$$E_S = \Delta_0, \qquad E_A = -\Delta_0, \qquad E_3 = 0.$$

Expressing the initial state in terms of the eigenstates

$$|\psi_{(t=0)}\rangle = |2\rangle = \frac{1}{\sqrt{2}}(|S\rangle - |A\rangle),$$

we can evolve it in time

$$|\psi_t\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\Delta_0 t}|S\rangle - e^{i\Delta_0 t}|A\rangle\right) = \frac{1}{2}\left[e^{-i\Delta_0 t}\left(|1\rangle + |2\rangle\right) - e^{i\Delta_0 t}\left(|1\rangle - |2\rangle\right)\right]$$
$$= \frac{1}{2}\left[\left(e^{-i\Delta_0 t} - e^{i\Delta_0 t}\right)|1\rangle + \left(e^{-i\Delta_0 t} + e^{i\Delta_0 t}\right)|2\rangle\right] = -i\sin(\Delta_0 t)|1\rangle + \cos(\Delta_0 t)|2\rangle$$

and read off the survaival amplitude for staying at site 2,

$$\psi_2(t) = \langle 2|\psi_t\rangle = \cos(\Delta_0 t).$$

(3) The pertubtion $c \neq 0$ is now turned on. To find $\frac{\partial \psi_3(t)}{\partial t}$, we use Eq'(591) from the lecture notes

$$\frac{\partial \psi_3(t)}{\partial t} = E_3|3\rangle - i\sum_m V_{3m}\psi_m(t)$$

In our case, $E_3 = 0$ and

$$V_{nn'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & c & 0 \end{pmatrix}$$

thus

$$\frac{\partial \psi_3(t)}{\partial t} = -ic\psi_2(t) = -ic \cdot \cos(\Delta_0 t)$$
$$\implies \psi_3(t) = \psi_3(0) - \frac{ic}{\Delta_0}\sin(\Delta_0 t)$$

(4) For $c \neq 0$, the Hamiltonian is:

$$\hat{\mathcal{H}} = \begin{pmatrix} 0 & \Delta_0 & 0 \\ \Delta_0 & 0 & c \\ 0 & c & 0 \end{pmatrix}$$

The eigenstates are:

$$|\tilde{1}\rangle = \frac{1}{\sqrt{c^2+\Delta_0^2}}\begin{pmatrix} c \\ 0 \\ -\Delta_0 \end{pmatrix}, \quad |\tilde{2}\rangle = \frac{1}{\sqrt{2(c^2+\Delta_0^2)}}\begin{pmatrix} \Delta_0 \\ \sqrt{\Delta_0^2+c^2} \\ c \end{pmatrix}, \quad |\tilde{3}\rangle = \frac{1}{\sqrt{2(c^2+\Delta_0^2)}}\begin{pmatrix} -\Delta_0 \\ \sqrt{\Delta_0^2+c^2} \\ -c \end{pmatrix},$$

and the eigenvalues:

$$E_{\tilde{1}} = 0, \quad E_{\tilde{2}} = \sqrt{\Delta_0^2 + c^2}, \quad E_{\tilde{3}} = -\sqrt{\Delta_0^2 + c^2}.$$

(5) The particle was prepared in state $|2\rangle$. In terms of the eigenstates,

$$|2\rangle = \frac{\Delta_0}{\sqrt{2}c}\left(|\tilde{2}\rangle - |\tilde{3}\rangle\right).$$

The state is a superposition of $|\tilde{2}\rangle$ and $|\tilde{3}\rangle$. Since the intital state does not depend on the eigenstate $|\tilde{1}\rangle$, it is not relevant to the solution.

(6) To find the oscillation frequency from site to site we evolve the state in time,

$$|\psi_t\rangle = \frac{\Delta_0}{\sqrt{2}c}\left(e^{-iE_{\tilde{2}}t}|\tilde{2}\rangle - e^{-iE_{\tilde{3}}t}|\tilde{3}\rangle\right)$$

$$= \frac{1}{2\sqrt{\Delta_0^2+c^2}}\left[2i\Delta_0\text{sin}(\sqrt{\Delta_0^2+c^2}t)|1\rangle + 2\sqrt{\Delta_0^2+c^2}\text{cos}(\sqrt{\Delta_0^2+c^2}t)|2\rangle + 2ic\cdot\sin(\sqrt{\Delta_0^2+c^2}t)|3\rangle\right]$$

thus the frequency is $\omega = \sqrt{\Delta_0^2+c^2}$.

(7) To find the probability for being in site 3 we project on the state $|3\rangle$

$$P_t = |\langle 3|\psi_t\rangle|^2 = \frac{c^2}{\Delta_0^2+c^2}\sin^2(\sqrt{\Delta_0^2+c^2}t).$$