Ex1391: particle in 4 sites system

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The problem:

Given 4 sites system, sites located at the vertex of a square rectangle. Transmission amplitude per time unit of charged particle between two adjacent sites is u. Homogeneous magnetic field vertical to the square's plan so the total flux is ϕ. Attached a diagonal conductor "+" between sites 1 and 3, and a diagonal conductor "-" between sites 2 and 4. Transmission amplitude per time unit through the oblique line is c. We will write the Hamiltonian $\mathcal{H}_F = \mathcal{H}_0 + V^+ + V^-$ and later we will refer to a system described by $\mathcal{H}_S = \mathcal{H}_0 + V^+$ At task (6) preparing the system with particle that it's momentum is zero, and check if there are oscillations (for example in current) as a result from the diagonal connections. You have to refer to $\mathcal{H}_F$ and $\mathcal{H}_S$ separately.

(1) Write $\mathcal{H}_0$ by using the momentum operator.
(2) Write V^+ and V^- in the position basis.
(3) Find the eigenstates and the eigenvalues of $\mathcal{H}_F$.
(4) Find the eigenstates of $\mathcal{H}_S$.
(5) How much we need to enlarge ϕ in task (4) in order to get the same energy spectrum.
(6) If there are oscillations in the system, write their frequencies.
Tip: You should rearrange the momentum basis so the blocks of V^+ and V^- will be clear.

The solution:

(1) All the diagonal elements and all the hopping amplitudes are the same, thus for 4 sites we can write:

$$\mathcal{H}_0 = u^- D + u^{-*} D^{-1} + const$$

We define $u^- = ue^{i\phi}$, where u is real, and get:

$$\mathcal{H}_0 = ue^{-i(p-\phi/4)} + ue^{i(p-\phi/4)} = 2ucos(p - \phi/4)$$

The eigenvalues for the translation operator are

$$k = \frac{2\pi n}{4} = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}.$$

(2) The momentum states are defined as follows:

$$|k\rangle = \Sigma \frac{1}{2} e^{ikx} |x\rangle$$

After reordering (correspondingly to $0, \pi, \frac{\pi}{2}, \frac{3\pi}{2}$) we get

$$T = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -i & i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i & -i \end{pmatrix}$$

In the standard base

$$V^+ = c\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix} \quad V^- = c\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{pmatrix}$$

In the momentum base we get

$$V^+ = \frac{c}{2}\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&-1&-1\\0&0&-1&-1\end{pmatrix} \quad V^- = \frac{c}{2}\begin{pmatrix}1&-1&0&0\\-1&1&0&0\\0&0&-1&1\\0&0&1&-1\end{pmatrix}$$

(3) In the momentum base $\mathcal{H}_0$ is diagonal.

$$\mathcal{H}_F = \mathcal{H}_0 + V^+ + V^-$$

$\mathcal{H}_F$ is diagonal and the eigenvalues are:

$$E_1 = 2ucos(\frac{\phi}{4}) + c \quad |E_1\rangle = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

$$E_2 = -2ucos(\frac{\phi}{4}) + c \quad |E_2\rangle = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$$

$$E_3 = 2usin(\frac{\phi}{4}) - c \quad |E_3\rangle = \frac{1}{2}\begin{pmatrix}1\\-i\\-1\\i\end{pmatrix}$$

$$E_4 = -2usin(\frac{\phi}{4}) - c \quad |E_4\rangle = \frac{1}{2}\begin{pmatrix}1\\i\\-1\\-i\end{pmatrix}$$

(4)

$$\mathcal{H}_S = \mathcal{H}_0 + V^+ = \begin{pmatrix}2ucos(\frac{\phi}{4}) + \frac{c}{2} & \frac{c}{2} & 0 & 0\\ \frac{c}{2} & -2ucos(\frac{\phi}{4}) + \frac{c}{2} & 0 & 0\\ 0 & 0 & 2usin(\frac{\phi}{4}) - \frac{c}{2} & -\frac{c}{2}\\ 0 & 0 & -\frac{c}{2} & -2usin(\frac{\phi}{4}) - \frac{c}{2}\end{pmatrix}$$

As we can see $\mathcal{H}_S$ is block diagonal, by calculating directly(2 Matrix(2x2)) we get

$$E_1 = \frac{c}{2} + \sqrt{(\frac{c}{2})^2 + 4u^2cos^2\frac{\phi}{4}}$$

$$E_2 = \frac{c}{2} - \sqrt{(\frac{c}{2})^2 + 4u^2cos^2\frac{\phi}{4}}$$

$$E_3 = -\frac{c}{2} + \sqrt{(\frac{c}{2})^2 + 4u^2 sin^2\frac{\phi}{4}}$$

$$E_4 = -\frac{c}{2} - \sqrt{(\frac{c}{2})^2 + 4u^2 sin^2\frac{\phi}{4}}$$

(5)
There are two cases, the first refers to the coupled states. One can see that the energy spectrum is periodic by 4π. The second case refers to the uncoupled states ($c = 0$). One can see that the energy spectrum in this case is periodic by 2π.

(6)
The system prepared in k = 0 momentum state. k = 0 is eigenstate of $\mathcal{H}_F$ so the system doesn't oscillate. In the case of $\mathcal{H}_S$, the matrix is block diagonal so we know that if the particle prepared in k = 0 momentum state than the system does oscillations between k = 0 or k = π momentum states, thus we get:

$$\Omega = E_1 - E_2 = 2\sqrt{(\frac{c}{2})^2 + 4u^2 cos^2\frac{\phi}{4}}$$