E137: 3 site system with time depended perturbation

Submitted by: Nimrod Beit Halachmi

The problem:

A particle can be found in one of three locations organized in the form of triangular ring. We define a space operator which can have the values: $\hat{x} = -1, 0, 1$, and a reflection operator $\hat{R}$ so that $|x\rangle \mapsto |-x\rangle$.

(a) Write the matrix representation of $\hat{x}$ and $\hat{R}$.

The Hamiltonian H which describes the particle's motion is

$$H = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

(b) Write the eigenstates $|k\rangle$ and the eigenvalue E_k of the system, where k is the momentum.

(c) Write an alternative eigenstates basis with defined symmetry (symmetric and anti-symmetric with respect to $\hat{R}$).

(Use the notation $|S\rangle$ and $|A\rangle$ for the high energy states).
A particle is prepared in the ground state $|S0\rangle$.
A pulse is applied $f(t) = \epsilon \mathrm{e}^{-\frac{1}{2}\left(\frac{t}{\tau}\right)^2}$ so the total Hamiltonian is $H_{total} = H + f(t) V(x)$.

In the following use first order perturbation theory in the calculations.

(d) What is the probability of finding the particle in the state $|S\rangle$, after the pulse ended, if $V(x) = x$?

(e) What is the probability of finding the particle in the state $|S\rangle$, after the pulse ended, if $V(x) = x^2$?

The solution:

(a) The matrix representation of the operator $\hat{x}$ in the position basis is:

$$\hat{x} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix representation of the operator $\hat{R}$ in the position basis is:

$$\hat{R} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(b) The Hamiltonian can be written as $H = -\left(\hat{D} + \hat{D}^{\dagger}\right)$ so that the momentum states are the eigenstates of the Hamiltonian. We can expand the momentum states in the position basis as:

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_x \mathrm{e}^{ikx} |x\rangle \qquad , \qquad k = \frac{2\pi n}{N}$$

In our case

$$k_{0,1,2} = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

The eigenstates are:

$$|k_0\rangle = \frac{1}{\sqrt{3}}\left(|-1\rangle + |0\rangle + |1\rangle\right)$$

$$|k_1\rangle = \frac{1}{\sqrt{3}}\left(\mathrm{e}^{\frac{-i2\pi}{3}}|-1\rangle + |0\rangle + \mathrm{e}^{\frac{i2\pi}{3}}|1\rangle\right)$$

$$|k_2\rangle = \frac{1}{\sqrt{3}}\left(\mathrm{e}^{\frac{i2\pi}{3}}|-1\rangle + |0\rangle + \mathrm{e}^{\frac{-i2\pi}{3}}|1\rangle\right)$$

In order to find the Eigenenergies we use the fact that the Hamiltonian is a combination of displacment operators so the following equality holds

$$H|k\rangle = -\left(\hat{D} + \hat{D}^{\dagger}\right)|k\rangle = -\left(\mathrm{e}^{-ik} + \mathrm{e}^{ik}\right)|k\rangle = -2cosk\,|k\rangle$$

Therfore the eigenenergies are:
The ground state $E_0 = -2$, the high energy states $E_1 = E_2 = 1$.

(c) The alternative basis with defined symmetry with respect to $\hat{R}$ is

$$|S0\rangle = |k_0\rangle = \frac{1}{\sqrt{3}}\left(|-1\rangle + |0\rangle + |1\rangle\right)$$

$$|S\rangle = \frac{1}{\sqrt{2}}\left(|k_1\rangle + |k_2\rangle\right) = \frac{1}{\sqrt{6}}\left(|-1\rangle - 2|0\rangle + |1\rangle\right)$$

$$|A\rangle = \frac{1}{\sqrt{2}}\left(|k_1\rangle - |k_2\rangle\right) = \frac{1}{\sqrt{2}}\left(|-1\rangle - |1\rangle\right)$$

(d)+(e) The first order time depended perturbation theory states that the probability to find a particle, that was prepered in the $|m\rangle$ state, in the $|n\rangle$ state after the perturbation is

$$P_{n,m} = |V_{n,m}|^2\,|FT[f(t)]|^2$$

The Fourier Transform of the perturbation is

$$FT[f(t)] = \epsilon\sqrt{2\pi}\tau\mathrm{e}^{-\frac{1}{2}(\tau\omega)^2} \qquad ,\omega = E_S - E_{S0} = 3$$

In the case of $V(x) = \hat{x}$ we get

$$V_{S,S0} = \langle S|\,\hat{x}\,|S0\rangle = 0 \qquad \Longrightarrow \qquad P_{S,S0} = 0$$

In the case of $V(x) = \hat{x}^2$ we get

$$V_{S,S0} = \langle S|\,\hat{x}^2\,|S0\rangle = \frac{1}{3\sqrt{2}}\begin{pmatrix}1 & -2 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix} = \frac{\sqrt{2}}{3}$$

and

$$P_{S,S0} = \frac{4\pi}{9}(\epsilon\tau)^2\,\mathrm{e}^{-(3\tau)^2}$$