E135: A particle in a three sites system, current

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The problem:

A particle can be found in one of three sites, organized as a triangle-ring. The position operator can have these values: $x = -1, 0, 1$. The Hamiltonian which describes the motion of the particle on the ring is defined in Eq. 1.

(1) Define the translation operator D in the simplest way. Also write D^{-1}.

(2) Find the eigenstates $|k>$ and the eigen-energies E_k of the system, where k is the momentom.

A Magnetic flux of Φ through the ring is added to the system. In the appropriate calibration the Hamiltion is define in Eq. 2.

(3) Define the value of c, when the charge of the particle e is given.

(4) Write the Hamiltonian in terms of the translation operator.

(5) Write the eigen-eneregies E_k.

(6) Write the electric current I of the particle when $k = 0$.

Eq. 1: $\mathcal{H} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$

Eq. 2: $\mathcal{H} = \begin{pmatrix} 0 & c* & c \\ c & 0 & c* \\ c* & c & 0 \end{pmatrix}$

The solution:

(1) The translation operator in the position basis is

$$\mathcal{D} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathcal{D}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

(2)

$$k = \frac{2\pi n}{N} = \frac{2\pi n}{3} \Rightarrow k_- = -\frac{2\pi}{3}, \quad k_0 = 0, \quad k_+ = \frac{2\pi}{3},$$

$$|k >= \Sigma \frac{1}{\sqrt{N}} e^{ikx} |x >$$

Therefore:

$$|-_k>= \frac{1}{\sqrt{3}}(e^{\frac{2\pi i}{3}}, 1, e^{-\frac{2\pi i}{3}})$$

$$|0_k>= \frac{1}{\sqrt{3}}(1, 1, 1)$$

$$|+_k>= \frac{1}{\sqrt{3}}(e^{-\frac{2\pi i}{3}}, 1, e^{\frac{2\pi i}{3}})$$

The eigenstates of the Hamiltonian are in the position basis. Since the system is symmetric under translations, the eigenstates of the Momentum(translation) operator and the Hamiltonian are the same.

$$H|k>= E_k|k>$$

$$\Downarrow$$

$$E_\pm = -2cos(\frac{2\pi}{3})$$

$$E_0 = -2$$

(3) The flux will add a phase to the Hemiltonian of Eq. 1, meaning, the mass of the particle stays the same.
The total phase the particle would accumulate in surrounding the whole ring is $\phi = e\Phi$. We assume the distanse between any two sites is $\frac{1}{3}$ of the ring's circumference, therefore, $\phi = \frac{e\Phi}{3}$ and $c = -e^{i\phi}$.

(4)

$$H = cD + c^* D^{-1}$$

(5) By using the same method as in 2 (The system is still symmetric under translations).

$$E_- = -2cos(\frac{2\pi}{3} + \phi)$$

$$E_+ = -2cos(\frac{2\pi}{3} - \phi)$$

$$E_0 = -2cos(\phi)$$

(6) By using $I = -\frac{dE_0}{d\Phi}$, We get:

$$I = -2\sin(\frac{e\Phi}{3}) \cdot \frac{e}{3}$$