Ex1334: Eigenstates of a network with a rank-1 perturbation

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The problem:

Consider Hamiltonian of the type $H_0 + \lambda\left(|u\rangle\langle v| + |v\rangle\langle u|\right)$. The rank-1 perturbation term generates jumps between the states u and v. Assume that the eigenstates n of H_0 are known, and that u_n and v_n are given.

(1) Write that Hamiltonian matrix H_{nm} and the corresponding eigen-equation.
(2) Show that the functional form of the eigenstates can be found explicitly in terms of two parameters.
(3) Write the reduced eigen-equation for the two parameters.
(4) Explain why the following models are special cases:
  (a) A network in which only two sites "1" and "2" are coupled;
  (b) A ring with a delta perturbation at some "0" site.
  (c) A site "0" that is coupled to a set of decoupled levels "k".
Write the simplified equation for the eigen-energies in the above cases.

The solution:

(1)

$$H = H_0 + \lambda[\sum_k |k\rangle\langle k|u\rangle \sum_l \langle v|l\rangle\langle l|] + \lambda[\sum_i |i\rangle\langle i|v\rangle \sum_j \langle u|j\rangle\langle j|]$$

$$H_{nm} = \lambda_n \delta_{nm} + \lambda[u_n v_m^* + v_n u_m^*]$$

(2) + (3)

$$H_{n,m}\psi_m = E\psi_n$$

$$\epsilon_n \psi_n + \lambda u_n \sum_m v_m^* \psi_m + \lambda v_n \sum_m u_m^* \psi_m = E\psi_n$$

we'll define two parameters:

$$C_v = \sum_m v_m^* \psi_m \qquad (I) \qquad\qquad C_u = \sum_m u_m^* \psi_m \qquad (II)$$

and get:

$$\psi_n = \frac{\lambda u_n C_v + \lambda v_n C_u}{(E - \epsilon_n)}$$

we put this back into (I),(II):

$$C_v = \lambda C_v \sum_m \frac{v_m^* u_m}{E - \epsilon_m} + \lambda C_u \sum_m \frac{v_m^* v_m}{E - \epsilon_m}$$

$$C_u = \lambda C_v \sum_m \frac{u_m^* u_m}{E - \epsilon_m} + \lambda C_u \sum_m \frac{u_m^* v_m}{E - \epsilon_m}$$

for these two equations to have a non-trivial solution we demand:

$$\begin{vmatrix} \lambda A_{v,u} - 1 & \lambda A_{v,v} \\ \lambda A_{u,u} & \lambda A_{u,v} - 1 \end{vmatrix} = 0$$

where

$$A_{v,v} = \sum_m \frac{|v_m|^2}{E - \epsilon_m} \qquad A_{u,u} = \sum_m \frac{|u_m|^2}{E - \epsilon_m}$$

$$A_{v,u} = \sum_m \frac{v_m^* u_m}{E - \epsilon_m} \qquad A_{u,v} = \sum_m \frac{u_m^* v_m}{E - \epsilon_m}$$

solving the determinant will give us a closed equation for the eigenvalues .

(4.a)

the hamiltonian takes on the form

$$H = H_0 + \lambda(|1\rangle\langle 2| + |2\rangle\langle 1|)$$

this gives:

$$u_n = \delta_{1,n} \qquad v_n = \delta_{2,n}$$

$$\begin{vmatrix} -1 & \frac{\lambda}{E-\epsilon_2} \\ \frac{\lambda}{E-\epsilon_1} & -1 \end{vmatrix} = 0$$

$$\lambda^2 = (E - \epsilon_1)(E - \epsilon_2)$$

(4.b)

the perturbation matrix takes on the form (lecture notes, page 214):

$$V = \lambda \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \cdots \\ \sqrt{2} & 2 & 2 & 2 & \cdots \\ \sqrt{2} & 2 & 2 & 2 & \cdots \\ \sqrt{2} & 2 & 2 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

this matrix is obviously a rank-one matrix, hence it may be written in the form of:

$$V_{n,m} = \lambda[u_n v_m^* + v_n u_m^*]$$

with:

$$u_1 = v_1 = \frac{1}{\sqrt{2}}$$

$$u_{n\neq 1} = v_{n\neq 1} = 1$$

therefore:

$$A_{v,u} = A_{u,v} = A_{v,v} = A_{u,u} = A$$

$$\begin{vmatrix} \lambda A - 1 & \lambda A \\ \lambda A & \lambda A - 1 \end{vmatrix} = 0$$

$$A = \sum_m \frac{u_m^2}{E - \epsilon_m} = \frac{1}{2\lambda}$$

$$\epsilon_m \propto m^2$$

(4.c)

the hamiltonian takes on the form:

$$H = H_0 + \sum_{n=1} \sigma_n (|n\rangle\langle 0| + |0\rangle\langle n|)$$

meanning:

$$u_0 = 0 \qquad\qquad v_0 = 1$$

$$u_{n\neq 0} = \sigma_n \qquad\qquad v_{n\neq 0} = 0$$

and therefore the non-zero A's are:

$$A_{v,v} = \frac{1}{E - \epsilon_0} \qquad\qquad A_{u,u} = \sum_m \frac{\sigma_m^2}{E - \epsilon_m}$$

form the determinant we get:

$$E - \epsilon_0 = \sum_m \frac{\sigma_m^2}{E - \epsilon_m}$$