Ex1324: A particle translating in a 4 site system

Submitted by: Moamen Jbara

This solution is an upgraded version of submission by Lavit Shachar

The problem:

The position of a particle is defined by the operator $\hat{x}$ that can get integer values only (modulo 4). $\hat{D}$ is defined as the displacement operator which move the particle one step counter-clock wise.

We define the following :

a) $\hat{x}|x\rangle = x|x\rangle$.
b) $\hat{D}|x\rangle = |x+1\rangle$.
c) e^{ik} - The eigenvalue of $\hat{D}$.
d) $|k\rangle$ - The eigenstate of $\hat{D}$.
e) The eigenstate of $\hat{D}$ represented by $\psi(x) = \langle x|k\rangle$.
f) The momentum operator is defined by $\hat{p}|k\rangle = k|k\rangle$.

(1) Write the matrix representation of the operators $\hat{x}$, $\hat{D}$ in the position basis.
(2) What are the possible values of k?
(3) Write an expression for the eigenvectors.
(4) Express the operator $\hat{D}$ using $\hat{p}$.
(5) Use the calibration freedom and write the most general Hamiltonian that fully describe the system's evolution.
(6) How many phases and parameters appear in the Hamiltonian?
(7) Answer the questions 5 and 6 again if its given that the only allowed translations are between neighboring sites.
(8) Assume also that there is translations symmetry and express the Hamiltonian using the $\hat{D}$ operator.
(9) For the last case-write an expression of $\hat{H}$ using $\hat{p}$.
(10) For the last case-write an expression for the eigenvalues (Assume no magnetic field).

The solution:

(1) By using definition (a), we can write the position matrix in the position basis as follows :

$$\hat{x} \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

For a closed system (periodic boundary conditions), we can calculate (by using definition (b)) the translation operator represented in the position basis :

$$\hat{D} \longmapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(2) The possible values of k are:

$k = \frac{2\pi}{4}n$ where n =integer mod(4).

That means that k can equals :

$k = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

(3) The eigenvectors for the operator $\hat{D}$ are:

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_x e^{ikx} |x\rangle$$

For our case, $N = 4$ which gives :

$$|k = 0\rangle = \frac{1}{2}(|1\rangle + |2\rangle + |3\rangle + |4\rangle)$$

$$|k = \frac{\pi}{2}\rangle = \frac{1}{2}(i|1\rangle - |2\rangle - i|3\rangle + |4\rangle)$$

$$|k = \pi\rangle = \frac{1}{2}(-|1\rangle + |2\rangle - |3\rangle + |4\rangle)$$

$$|k = \frac{3\pi}{2}\rangle = \frac{1}{2}(-i|1\rangle - |2\rangle + i|3\rangle + |4\rangle)$$

(4) According to (f), we know that : $\hat{p}|k\rangle = k|k\rangle$.

$\hat{D}|k\rangle = \hat{D}(\frac{1}{\sqrt{N}} \sum_x e^{ikx}|x\rangle) = \frac{1}{\sqrt{N}} \sum_x e^{ikx}|x{+}1\rangle = (y = x{+}1) = \frac{1}{\sqrt{N}} \sum_y e^{ik(y-1)}|y\rangle = e^{-ik} \frac{1}{\sqrt{N}} \sum_y e^{iky}|y\rangle = e^{-ik}|k\rangle$

The previous equation is right for every k, using the definition of "function of an operator" we can write $\hat{D}$ as:

$$\hat{D} = e^{-i\hat{p}}$$

(5) The most general Hamiltonian is:

$$\mathcal{H} = \begin{pmatrix} \epsilon_1 & c_1 e^{-i\phi_1} & c_2 e^{-i\phi_2} & c_3 e^{-i\phi_3} \\ c_1 e^{i\phi_1} & \epsilon_2 & c_4 e^{-i\phi_4} & c_5 e^{-i\phi_5} \\ c_2 e^{i\phi_2} & c_4 e^{i\phi_4} & \epsilon_3 & c_6 e^{-i\phi_6} \\ c_3 e^{i\phi_3} & c_5 e^{i\phi_5} & c_6 e^{i\phi_6} & \epsilon_4 \end{pmatrix}$$

The Hamiltonian above has 16 degrees of freedom, 4 degrees due to diagonal elements and 12 degrees due to the 6 complex off-diagonal elements (H is hermitian).

Without loss of generality we can set $\epsilon_1 = 0$.

Thanks to gauge freedom we can define a new basis:

$|1'\rangle = |1\rangle$
$|2'\rangle = e^{i\phi_1}|2\rangle$
$|3'\rangle = e^{i\phi_2}|3\rangle$
$|4'\rangle = e^{i\phi_3}|4\rangle$

Then the Hamiltonian can be written as:

$$\tilde{\mathcal{H}} = \begin{pmatrix} 0 & c_1 & c_2 & c_3 \\ c_1 & \epsilon_2 & c_4 e^{-i\tilde{\phi}_4} & c_5 e^{-i\tilde{\phi}_5} \\ c_2 & c_4 e^{i\tilde{\phi}_4} & \epsilon_3 & c_6 e^{-i\tilde{\phi}_6} \\ c_3 & c_5 e^{i\tilde{\phi}_5} & c_6 e^{i\tilde{\phi}_6} & \epsilon_4 \end{pmatrix}$$

The Hamiltonian has 3 phases (gauge invariants) because there are three different loops in our system.

(6) According to the explanation provided in solution of (5), the most general Hamiltonian for a particle in a 4 site system includes 9 parameters and 3 phases.

(7) In this case, the translations between the sites : 1-3, 2-4 are forbidden. Therefore the amplitudes c_2 and c_5
are zero. In addition, now we have only one loop (one gauge invariant) therefore we need only one phase in the Hamiltonian. So we can write the Hamiltonian as follows :

$$\tilde{\mathcal{H}} = \begin{pmatrix} 0 & c_1 & 0 & c_3 \\ c_1 & \epsilon_2 & c_4 & 0 \\ 0 & c_4 & \epsilon_3 & c_6 e^{-i\tilde{\phi}_6} \\ c_3 & 0 & c_6 e^{i\tilde{\phi}_6} & \epsilon_4 \end{pmatrix}$$

We need 7 parameters and 1 phase to describe the evolution of this system.

(8) Its given that the system has a translation symmetry, as a result we can deduce the following :

(a) $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0$
(b) All the transition amplitudes are equal and the phase is divided equally between the 4 sites, therefore we can write : $c = c_0 e^{i\Phi/4}$.

So we can write the Hamiltonian as follows :

$$\mathcal{H} = \begin{pmatrix} 0 & c^* & 0 & c \\ c & 0 & c^* & 0 \\ 0 & c & 0 & c^* \\ c^* & 0 & c & 0 \end{pmatrix} = c\hat{D} + c^*\hat{D}^{-1}$$

(9) According to (4) and (8), we know that $\hat{D} = e^{-i\hat{p}}$ and $c = c_0 e^{i\Phi/4}$.
If we substitute these two equations in the expression of the Hamiltonian we found in (8), we get : $\hat{H} = 2c_0 * \cos(\hat{p} - \Phi/4)$.

(10) Its given that there is no magnetic field so we can deduce that the total phase equals zero. Since the Hamiltonian is a function of $\hat{p}$, its eigenstates are the momentum states.

$\hat{H}|k\rangle = (2c_0 * \cos(\hat{p}))|k\rangle = (2c_0 * \cos(k))|k\rangle$.

As a result the eigenenergies will be (i = 1,2,3,4 numbers the eigenenergies, i=1 represent the lowest momentum value (k=0), the bigger number i is the higher the value of the momentum) :

$$E_{(1)} = 2c_0 * \cos(0) = 2c_0$$
$$E_{(2)} = 2c_0 * \cos(\frac{\pi}{2}) = 0$$
$$E_{(3)} = 2c_0 * \cos(\pi) = -2c_0$$
$$E_{(4)} = 2c_0 * \cos(\frac{3\pi}{2}) = 0$$