Ex132: A two sited system

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The problem:

The location of a partical is defined by Operator $\hat{x}$ and can get the values $x = 1, -1$ We define another Operator $\hat{D}$ that changes the location of the partical.

(1)Write the matrix presentation of both Operators using the base set by $\hat{x}$.
(2)Find the eigenstates of $\hat{D}$ and write them as $|S\rangle$ and $|A\rangle$.

The system is symmetrical. The hopping amplitude per unit time is c.

(3)Write the matrix presentation of the Hamiltonian $\mathcal{H}$.
(4)WHat are the stationary states of the system.

Assume that when $t = o$ the partical is in the right site $|-1\rangle$.

(5)Write the initial state as a superposition of the atationary states.
(6)Write the state of the partical at $t > 0$ (use the Dirac notation).
(7)Find the probability $P(t)$ of the partical to be in the right site $|-1\rangle$.
(8)Find $\langle x\rangle$ as a function of time.

The solution:

(1) The location of a partical is defined by the Operator $\hat{x}$ and can have the values x=1,-1.

$$\hat{x}|1\rangle = |1\rangle$$

$$\hat{x}|-1\rangle = -|-1\rangle$$

The vectors $|1\rangle$ and $|-1\rangle$ are unit vectors of the standart basis.

$$\hat{x} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{D}|1\rangle = |-1\rangle$$

$$\hat{D}|-1\rangle = |1\rangle$$

$$\hat{D} \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(2) The eigenstates for the Operator $\hat{D}$ are $|S\rangle$ and $|A\rangle$

$$|S\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |-1\rangle) \longmapsto \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|A\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |-1\rangle) \longmapsto \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

It can easely be shown that $\hat{D}|S\rangle = |S\rangle$ and $\hat{D}|A\rangle = -|A\rangle$

(3) Let us now add the fact that the system is symetrical and has the transition amplitude of c .

$$\langle -1|\mathcal{H}|1\rangle = c$$

$$\langle 1|\mathcal{H}|-1\rangle = c$$

The Hemiltonian $\mathcal{H}$ will then be:

$$\mathcal{H} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$$

(4) The stationary states of $\mathcal{H}$ are $|S\rangle$ and $|A\rangle$, and the eigenvalues are $E_S = c$ and $E_A = -c$

(5) We shall assume that at time t=0 the partical is at the right site $|-1\rangle$

$$|\psi_{(t=0)}\rangle = |-1\rangle = \frac{1}{\sqrt{2}}(|S\rangle - |A\rangle)$$

(6) To find $|\psi(t)\rangle$ we first need to find the evolution Operator of the Hmiltonian $\hat{U}(t)$.
$\hat{U}(t) = \exp[-\frac{\imath\mathcal{H}t}{\hbar}] \longmapsto \begin{pmatrix} e^{\frac{-\imath\mathcal{H}t}{\hbar}} & 0 \\ 0 & e^{\frac{\imath\mathcal{H}t}{\hbar}} \end{pmatrix}$ From this point on i will take $\hbar$ to be 1.
in Dirac notation:

$$\hat{U}(t) = |S\rangle e^{-\imath\mathcal{H}t}\langle S| + |A\rangle e^{\imath\mathcal{H}t}\langle A|$$

The state of the particle, after time t will be:

$$|\psi(t)\rangle = \hat{U}(t)|\psi_{(t=0)}\rangle = \frac{1}{\sqrt{2}}(e^{-\imath\mathcal{H}t}|S\rangle - e^{\imath\mathcal{H}t}|A\rangle) =$$

$$= \frac{1}{2}[(e^{-\imath\mathcal{H}t} - e^{\imath\mathcal{H}t})|1\rangle + (e^{-\imath\mathcal{H}t} + e^{\imath\mathcal{H}t})|-1\rangle] = -\imath\sin(ct)|1\rangle + \cos(ct)|-1\rangle$$

(7) The probability of finding the partical in the right site $|-1\rangle$ is:

$$P(t) = |\langle -1|\psi(t)\rangle|^2 = \cos^2(ct)$$

(8) By definition: $\langle x\rangle = \langle\psi(t)|x|\psi(t)\rangle$
From the previous questions we have $\hat{x} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\psi(t) \longmapsto \begin{pmatrix} -\imath\sin(ct) \\ \cos(ct) \end{pmatrix}$

$$\langle x\rangle = \sum_{ij}\psi_i^* x_{ij}\psi_j = |-\imath\sin(ct)|^2 - |\cos(ct)|^2 = \sin^2(ct) - \cos^2(ct) = -\cos(2ct)$$

NOTE: Knowing the eigenvalues of $\hat{x}$ (λ_1, λ_2), it is easy to see that when $|\psi\rangle = a|1\rangle + b|-1\rangle$ the expectation value will be $\langle x\rangle = \lambda_1|a|^2 + \lambda_2|b|^2$,
In our case: $\lambda_1 = 1$ and $\lambda_2 = -1$ so we get $\langle x\rangle = |a|^2 - |b|^2$