E124: Fourier transform of a triangle function

Submitted by: Idan Shilon

The problem:

Use the convolution theorem to find the Fourier transform of a triangle function out of the Fourier transform of a window function.

The solution:

Let us look at the convolution of two ordinary window functions, where the convolution operation defined as the following integral:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t - t^{'})g(t^{'})dt^{'}.$$

The window function is given as follows:

$$\Pi(t) = \begin{cases} 1, & |t| < a \\ 0, & else \end{cases}$$

Let us calculate the convolution of two window functions:

$$(\Pi_1 * \Pi_2)(t) = \int_{-\infty}^{\infty} \Pi_1(t - t^{'})\Pi_2(t^{'})dt^{'} = \begin{cases} 0, & t < -2a \\ \int_{-a}^{t+a} dt^{'} = t + 2a, & -2a < t < 0 \\ \int_{t-a}^{a} dt^{'} = -t + 2a, & 0 < t < 2a \\ 0, & t > 2a \end{cases}$$

Thus we get a triangle function:

$$(\Pi_1 * \Pi_2)(t) = \Lambda(t).$$

The convolution theorem claims that

$$\text{F.T}[f * g] = \text{F.T}[f] \cdot \text{F.T}[g],$$

and so we can calculate the Fourier transform of a triangle function. Using the convolution theorem and the result of exercise 123 we get:

$$\text{F.T}[\Lambda(t)] = \text{F.T}[(\Pi_1 * \Pi_2)(t)] = \text{F.T}[\Pi_1(t)] \cdot \text{F.T}[\Pi_2(t)] = 4a^2\text{sinc}^2(\omega a)$$