E123: Fourier transform of a derivative

Submitted by: Idan Shilon

The problem:

Given the Fourier transform of a general function, find the Fourier transform of its derivative. Use this result to find the Fourier transform of a window function out of the Fourier transform of an antisymmetric pair of delta functions.

The solution:

We are given the following:

$$\mathrm{F.T}[f(t)] = \mathrm{F}(\omega),$$

and we take into account that:

$$\lim_{t\to\pm\infty} f(t) \to 0.$$

We begin by writing explicitly:

$$\mathrm{F.T}[f^{'}(t)] = \int_{-\infty}^{\infty} f^{'}(t)\mathrm{e}^{i\omega t}dt.$$

Integration by parts gives us:

$$f(t)\mathrm{e}^{i\omega t}|_{-\infty}^{+\infty} - i\omega\int_{-\infty}^{\infty} f(t)\mathrm{e}^{i\omega t}dt = -i\omega\mathrm{F}(\omega),$$

and we get:

$$\mathrm{F.T}[f^{'}(t)] = -i\omega\mathrm{F}(\omega).$$

Let us represent a window function in the region $[-a, a]$ as a sum of two step functions:

$$\Pi(t) = \Theta(t+a) - \Theta(t-a).$$

We also note that

$$\delta(t+a) = \frac{d}{dt}\Theta(t+a).$$

Now, using what we have derived earlier we find:

$$\mathrm{F.T}[\delta(t+a) - \delta(t-a)] = \mathrm{F.T}[\frac{d}{dt}\Theta(t+a) - \frac{d}{dt}\Theta(t-a)] = -i\omega\mathrm{F.T}[\Theta(t+a) - \Theta(t-a)] = -i\omega\mathrm{F.T}[\Pi(t)],$$

and,

$$\mathrm{F.T}[\delta(t+a) - \delta(t-a)] = \int_{-\infty}^{\infty}\delta(t+a)\mathrm{e}^{i\omega t}dt - \int_{-\infty}^{\infty}\delta(t-a)\mathrm{e}^{i\omega t}dt = \mathrm{e}^{-i\omega a} - \mathrm{e}^{i\omega a} = -2i\sin(\omega a).$$

$$\Rightarrow \mathrm{F.T}[\Pi(t)] = 2a\mathrm{sinc}(\omega a)$$