Ex121

Submitted by: Eyal Shafran

The problem:

(a) Find the F.T. $f(\omega)$ of a step function $\Theta(t)$.

(b) Find the F.T. of a sgn(t) function.

(c) Calculate the inverse F.T. of $f(\omega)$ by closing the integration contour in the complex plane.

(d) Calculate the inverse F.T. of $f(\omega)$ by elementary integration (separating the main part)

The solution:

(a) Recall the elementary integral:

$$\int_0^\infty \exp -\lambda t = \frac{1}{\lambda}$$

$$f(\omega) = FT[\Theta(t)] = \int\limits_{-\infty}^{\infty} \Theta(t)\exp(i\omega t)dt = \int\limits_{0}^{\infty} \exp(i\omega t)dt$$

$$\omega = \omega' + i\epsilon, \epsilon > 0$$

$$\int\limits_{0}^{\infty} \exp(i\omega t)dt = \frac{1}{-i(\omega' + i\epsilon)} = \frac{i}{\omega' + i\epsilon}$$

For the limit $\epsilon \to 0$, $\omega' = w$ and our final solution is:

$$f(\omega) = FT[\Theta(t)] = \frac{i}{(\omega + i\epsilon)}.$$

(b)

$$sgn(t) = 2\Theta(t) - 1$$

$$FT[sgn(t)] = FT[(2\Theta(t) - 1] = 2FT[\Theta(t)] - FT[1]$$

$$\frac{i}{(\omega + i\epsilon)} = \frac{i(\omega - i\epsilon)}{\omega^2 + \epsilon^2} = \frac{i\omega}{\omega^2 + \epsilon^2} + \frac{\epsilon}{\omega^2 + \epsilon^2} = \frac{i}{\omega} + \pi\delta(\omega)$$

$$FT[sgn(t)] = \frac{2i}{\omega} + 2\pi\delta(\omega) - 2\pi\delta(\omega) = \frac{2i}{\omega}$$

(c)

$$f(t) = \int_{-\infty}^{\infty} \frac{f(\omega)}{2\pi} \exp(-i\omega t) dw = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(\omega + i\epsilon)} \exp(-i\omega t) dw = \frac{i}{2\pi} \lim_{R\to\infty} \int_{\Gamma} \frac{1}{(z + i\epsilon)} \exp(-izt) dz$$

Let us examine the contour (Γ) for our case: When $t > 0$ we close the contour in the bottom half plane, and since we are summing in a clockwise direction a factor of -1 must be taken into account. When $t < 0$ we close the contour in the upper half plane.

Using the Residue Theorem we know that:

$$\int_{\Gamma} \frac{1}{(z + i\epsilon)} \exp(-izt) dz = 2\pi i \sum_j Res(f, a_j)$$

where a_j stands for a pole of the integrand.

Notice we have one pole at $z = -i\epsilon$.
There are no poles enclosed in the upper half plane and therefore for $t < 0$ the integral yeilds 0.

$$Res(z = -i\epsilon) = \lim_{z\to -i\epsilon} (z + i\epsilon) \frac{1}{(z + i\epsilon)} \exp(-izt) = \exp(\epsilon t) \approx 1$$

Therefore when $t > 0$:

$$f(t) = (\frac{-i}{2\pi}) 2\pi i Res(z = -i\epsilon) = 1$$

And our final solution is:

$$f(t) = \Theta(t)$$

(d) In part b we saw the equality:

$$f(\omega) = \frac{i}{\omega + i\epsilon} = \frac{i}{\omega} + \pi\delta(\omega)$$

$$FT[f(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i}{\omega} \exp(-i\omega t) d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi\delta(\omega) \exp(-i\omega t) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i}{\omega} \exp(-i\omega t) d\omega + \frac{1}{2} =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i\cos(\omega t)}{\omega} d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin(\omega t)}{\omega} d\omega + \frac{1}{2}$$

$\frac{\cos(\omega t)}{w}$ is an odd function and therfore the first integral on the right side falls.

Recall the elementary integral :

$$\int_{-\infty}^{\infty} \frac{sin(x)}{x} dx = \pi$$

So our final solution is:

$$f(t) = \frac{1}{2} sgn(t) + \frac{1}{2} = \Theta(t)$$