E120: Fourier Transform of simple functions

Submitted by: Lior Shachaf

The problem:

Find the Fourier Transform of:

(1) Gaussian and Lorentzian.
(2) Shifted δ function.
(3) 2 "Cracks" represented by 2 δ functions.
(4) Prove $\sum e = \sum \delta()$

- direct
- using Poisson summation formula

(5) 1D lattice (a "comb").
(6) 2D square lattice composed from δ functions.

The solution:

We will use the physicist convention for fourier transform $\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-ikx}dx$ and the inverse transform $f(x) = \int_{-\infty}^{\infty} \hat{f}(k)\mathrm{e}^{ikx}\frac{dk}{2\pi}$

(1) For a Gaussian $\mathrm{e}^{-\frac{1}{2}(\frac{x}{\sigma})^2}$:

$$FT\left[\mathrm{e}^{-\frac{1}{2}(\frac{x}{\sigma})^2}\right] = \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}(\frac{x}{\sigma})^2}\mathrm{e}^{-ikx}dx = \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2\sigma^2}(x+ik\sigma^2)^2-\frac{1}{2}(\sigma k)^2}dx = \sqrt{2}|\sigma|\mathrm{e}^{-\frac{1}{2}(\sigma k)^2}\sqrt{\pi} = \sqrt{2\pi}|\sigma|\mathrm{e}^{-\frac{1}{2}(\sigma k)^2}$$

For a Lorentzian $L(x) = \frac{1}{\pi}\frac{\gamma}{x^2+\gamma^2}$ we get:

$$FT\left[\frac{1}{\pi}\frac{\gamma}{x^2+\gamma^2}\right] = \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{\gamma}{x^2+\gamma^2}\mathrm{e}^{-ikx}dx = \frac{\gamma}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-ikx}}{(x+i\gamma)(x-i\gamma)}dx$$

Now we will use Cauchy's integral formula. Although we only need to solve our integral for the real axis, we will have to use the formula once for each pole, because, if we decide to close the contour, only on the upper part of the imaginary axis, the integral on the arc will "explode" for $k > 0$.

For $k < 0$ we will close the contour, counter clockwise, on the upper plane, and use Cauchy's formula:

$$\frac{\gamma}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-ikx}}{(x+i\gamma)(x-i\gamma)}dx = \frac{\gamma}{\pi}\oint \frac{\frac{\mathrm{e}^{-izk}}{z+i\gamma}}{z-i\gamma}dz = \frac{2i\pi\gamma}{\pi}\frac{\mathrm{e}^{-i(i\gamma)k}}{2i\gamma} = \mathrm{e}^{\gamma k}$$

For $k > 0$ we will close the contour, clockwise, on the lower plane, not forgetting to add (-) to the integral:

$$\frac{\gamma}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ikx}}{(x+i\gamma)(x-i\gamma)}dx = \frac{\gamma}{\pi}\oint \frac{\frac{\mathrm{e}^{izk}}{z-i\gamma}}{z+i\gamma}dz = -\frac{2i\pi\gamma}{\pi}\frac{\mathrm{e}^{i(-i\gamma)k}}{-2i\gamma} = \mathrm{e}^{-\gamma k}$$

And so, we finally get:

$$FT\left[\frac{1}{\pi}\frac{\gamma}{x^2+\gamma^2}\right] = \mathrm{e}^{-\gamma|k|}$$

(2)

$$FT\left[\delta(x-a)\right] = \int_{-\infty}^{\infty} \delta(x-a)\mathrm{e}^{-ikx}dx = \mathrm{e}^{-ika}$$

(3)

$$FT\left[\delta(x-a)+\delta(x+a)\right] = \int_{-\infty}^{\infty} (\delta(x-a)+\delta(x+a))\mathrm{e}^{-ikx}dx =$$

$$= \mathrm{e}^{-ika} + \mathrm{e}^{ika} = 2cos(ka)$$

(4) (a) Direct proof: We will now show that the partial sum $F_N(x) = \sum_{k=-N}^{N} \mathrm{e}^{2\pi ikx}$ is equivalent to $\sum_{n=-\infty}^{\infty} \delta(x-n)$ as $N \to \infty$.
It is clear that $F_N(x)$ is periodic with period of 1, and so for ease of use we will consider only the range $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

$$F_N(x) = \sum_{k=-N}^{N} \mathrm{e}^{2\pi ikx} = \mathrm{e}^{-2\pi iNx} \sum_{m=0}^{2N} (\mathrm{e}^{2\pi ix})^m = \mathrm{e}^{-2\pi iNx} \frac{\mathrm{e}^{2\pi i(2N+1)x} - 1}{\mathrm{e}^{2\pi ix} - 1} =$$

$$= \mathrm{e}^{-2\pi iNx} \frac{\mathrm{e}^{2\pi i(2N+1)x}\left[\mathrm{e}^{\pi i(2N+1)x} - \mathrm{e}^{-\pi i(2N+1)x}\right]}{\mathrm{e}^{2\pi ix}\left[\mathrm{e}^{\pi ix} - \mathrm{e}^{-\pi ix}\right]} = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}$$

To see how $F_N(x)$ behaves like a δ distribution in the limit $N \to \infty$, we consider

$$\int_{-1/2}^{1/2} F_N(x)dx = 2\int_{0}^{1/2} \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}dx.$$

We make the substitution

$$u = (2N+1)\pi x,\ dx = \frac{du}{(2N+1)\pi},$$

so that

$$\int_{-1/2}^{1/2} F_N(x)dx = 2\int_{0}^{(N+1/2)\pi} \frac{\sin(u)}{(2N+1)\pi\sin(u/(2N+1))}du.$$

In the limit as $N \to \infty$, the upper limit of the integral goes to infinity, while the denominator of the integrand approches πu. The result is

$$\int_{-1/2}^{1/2} F_N(x)dx \xrightarrow{N\to\infty} \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin(u)}{u}du = \frac{2}{\pi}\frac{\pi}{2} = 1$$

In order to show the second quality of the δ function, we again consider the prior derivation of the integral over one period, but multiplied by a well-behaved function $f(x)$;

$$\int_{-1/2}^{1/2} f(x)F_N(x)dx = 2\int_{0}^{1/2} f(x)\frac{\sin((2N+1)\pi x)}{\sin(\pi x)}dx =$$

$$= 2\int_{0}^{(N+1/2)\pi} f(u/(2N+1)\pi)\frac{\sin(u)}{(2N+1)\pi\sin(u/(2N+1))}du$$

$$\rightarrow \frac{2}{\pi}\int_0^\infty f(0)\frac{\sin(u)}{u}du = f(0)\frac{2}{\pi}\int_0^\infty \frac{\sin(u)}{u}du = f(0),$$

where the same substitution, $u = (2N+1)\pi x$ has been made. This shows that

$$\lim_{N\to\infty}\sum \mathrm{e} = \sum \delta$$

(b) The Poisson summation formula is an equation relating the Fourier series coefficients of the periodic summation of a function to values of the function's continuous Fourier transform. In the general case:

$$\sum_{n=-\infty}^{\infty} f(x+na) = \frac{1}{a}\sum_{k=-\infty}^{\infty}\hat{f}(\frac{k}{a})\mathrm{e}^{2\pi i\frac{k}{a}x}$$

In particular, when $x = 0$ it is known as the Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(na) = \frac{1}{a}\sum_{k=-\infty}^{\infty}\hat{f}(\frac{k}{a})$$

And in our case, using Poisson:

$$\sum_{n=-\infty}^{\infty}\delta(x-na) = \frac{1}{a}\sum_{k=-\infty}^{\infty} FT\left[\delta(x-na)\right]_{(\frac{k}{a})} = \frac{1}{a}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-na)\mathrm{e}^{-i\frac{k}{a}x}dx = \frac{1}{a}\sum_{k=-\infty}^{\infty}\mathrm{e}^{-ikn}$$

(5) We define a 1D lattice as: $Comb(x) = \sum_{n=-\infty}^{\infty}\delta(x-na)$

This presentation is also known as the Dirac Comb function.
We will use the relation from the item above for the fourier transform of the comb function;

$$FT\left[\sum_{n=-\infty}^{\infty}\delta(x-na)\right] = FT\left[\frac{1}{a}\sum_{m=-\infty}^{\infty}\mathrm{e}^{i2\pi mx/a}\right] = \frac{1}{a}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\mathrm{e}^{i2\pi mx/a}\mathrm{e}^{-ikx}dx =$$

$$\frac{1}{a}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-ix(k-\frac{2\pi}{a}m)}dx = \frac{1}{a}\sum_{m=-\infty}^{\infty}\delta(k-\frac{2\pi}{a}m)$$

And so we get that the fourier transform of a 1D lattice is a 1D lattice with a reciprocal constant.

(6) Using the result from item 4;

$$FT\left[\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-nL_x)\delta(y-mL_y)\right] =$$

$$= \frac{1}{L_x}\frac{1}{L_y}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\mathrm{e}^{i2\pi nx/L_x}\mathrm{e}^{i2\pi my/L_y}\mathrm{e}^{-ik_xx}\mathrm{e}^{-ik_yy}dxdy =$$

$$= \frac{1}{L_xL_y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(k_x-\frac{2\pi}{L_x}n)\delta(k_y-\frac{2\pi}{L_y}m)$$