E108: The Displacement Operator

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The problem:

1. Write the displacement operator for N=3 site system.
2. Find the eigenvalues from the characteristic equation.
3. Find a set of eigenvectors.
4. Write the displacement operator for N=4 site system.
5. Write the eigenvalues k_n using results of the lecture.
6. Write explicitly what are the corresponding eigenvectors.

The solution:

(1) The displacement operator for the N=3 site system is:

$$\mathrm{D} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(2) To find the eigenvalues of $\hat{D}$ we first find the characteristic equation:

$$det(\hat{D} - \hat{I}\lambda) = 0$$

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

which gives:

$$\lambda^3 = 1$$

and therefore the eigenvalues are:

$$\lambda_n = 1, \mathrm{e}^{2\pi i/3}, \mathrm{e}^{4\pi i/3}$$

(3) The eigenvectors are:

$$\begin{pmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

the general eigenvalue is $\lambda_n = \mathrm{e}^{2\pi i n/3}$. We have one independent parameter so we can choose x to be 1 in order to get:

$$x = 1, y = \mathrm{e}^{-2\pi i n/3}, z = \mathrm{e}^{-4\pi i n/3}$$

The general eigenvector is:

$$\vec{\Psi}_x^{(n)} = \mathrm{e}^{-2\pi i n x/3}$$

The normalized vectors are:

$$\vec{\Psi}_x^{(n=0)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \vec{\Psi}_x^{(n=1)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \mathrm{e}^{4\pi i/3} \\ \mathrm{e}^{2\pi i/3} \end{pmatrix}, \vec{\Psi}_x^{(n=2)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \mathrm{e}^{2\pi i/3} \\ \mathrm{e}^{4\pi i/3} \end{pmatrix}$$

(4) The displacement operator for the $N = 4$ site system is:

$$\mathrm{D} \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(5)The base in which the displacment operator in diagonal is the momentum base.

In general:

$$\mathrm{D} \mid k\rangle = \mathrm{e}^{-ik} \mid k\rangle$$

Where

$$\mathrm{k} = \frac{2\pi n}{N}$$

Therfore the general eigenvalue is:

$$\lambda_n = \mathrm{e}^{-i\pi n/2}$$

Which gives:

$$\lambda_n = 1, \mathrm{e}^{\pi i/2}, \mathrm{e}^{\pi i}, \mathrm{e}^{3\pi i/2}$$

(6) The normalized eigenvectors are:

$$\vec{\Psi}_x^{(n)} = \frac{1}{\sqrt{4}} \mathrm{e}^{\pi i n x/2}$$

Therefore we get:

$$\vec{\Psi}_x^{(n=0)} = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \vec{\Psi}_x^{(n=1)} = \frac{1}{\sqrt{4}} \begin{pmatrix} i \\ -1 \\ -i \\ 1 \end{pmatrix}, \vec{\Psi}_x^{(n=2)} = \frac{1}{\sqrt{4}} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \vec{\Psi}_x^{(n=3)} = \frac{1}{\sqrt{4}} \begin{pmatrix} -i \\ -1 \\ i \\ 1 \end{pmatrix}$$