# E786: Calculation of the cross section for the Yukawa potential

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**The problem:**

Find the differential cross section and the total cross section by the Born approximation for Yukawa potential : $U(r) = U_0 \frac{e^{-(\frac{r}{a})}}{r}$ . Define the conditions for the use of the Born approximation.

**The solution:**

From the Fermi golden rule we have the formula for the differential cross section called the Born approximation

$$\sigma(\Omega) = \frac{1}{2\pi^2}\left(\frac{k_E}{v_E}\right)^2 \left|\tilde{U}\left(\vec{k_\Omega} - \vec{k_0}\right)\right|^2$$

In order to use the Born formula we define the system coordinates:
The incident wave propagates in the z direction. The scattering direction is $\Omega = (\theta, \varphi)$. We also define $\vec{q} = \vec{k_\Omega} - \vec{k_0}$ as the difference between the $k_\Omega$ of the scattered wave and the $k_0$ of the incident wave.
$k_E = \left|\vec{k_\Omega}\right| = \left|\vec{k_0}\right|$ (ellastic scattering).

The approximation is that the scattering potential can be treated as a perturbation, allowing us to use the Fermi golden rule, from which we get the formula above.

$\tilde{U}(\vec{q})$ is the Fourier transform of the scattering potential U(r):

$$\tilde{U}(\vec{q}) = \int\int\int U(r) e^{-i\vec{q}\vec{r}} d^3r = U_0 \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{\infty} dr \cdot r^2 sin\theta e^{-i\vec{q}\vec{r}cos\theta} \frac{e^{-\frac{r}{a}}}{r}$$

The given potential is spherically symmetric, so we can rotate coordinate system for the calculation of this integral. We choose $\vec{q}$ in the z direction, which means $\theta = 0$.

$$\tilde{U}(\vec{q}) = 2\pi U_0 \int_0^{\infty} dr \cdot r \cdot e^{-\frac{r}{a}} \int_{-1}^{1} d[cos\theta] e^{-iqrcos\theta} =$$

$$2\pi U_0 \int_0^{\infty} dr \cdot r \cdot e^{-\frac{r}{a}} \frac{e^{-iqrcos\theta}}{-iqr} |_{-1}^{1} = 2\pi U_0 \int_0^{\infty} dr \cdot e^{-\frac{r}{a}} \frac{e^{-iqr} - e^{iqr}}{-iq} =$$

$$\frac{2\pi U_0}{iq} \int_0^{\infty} dr \left[e^{\left(iq - \frac{1}{a}\right)r} - e^{\left(-iq - \frac{1}{a}\right)r}\right] = \frac{2\pi U_0}{iq}\left[\frac{e^{\left(iq-\frac{1}{a}\right)r}}{iq - \frac{1}{a}} - \frac{e^{\left(-iq-\frac{1}{a}\right)r}}{-iq-\frac{1}{a}}\right] |_0^{\infty} =$$

$$\frac{2\pi U_0}{iq}\left[\frac{-1}{iq-\frac{1}{a}} + \frac{-1}{iq+\frac{1}{a}}\right] = \frac{2\pi U_0}{iq} \frac{2iq}{q^2 + \frac{1}{a^2}} =$$

$$\frac{4\pi U_0}{q^2 + \frac{1}{a^2}}$$

$\frac{k_E}{v_E} = m$ (non-relativistic dispersion relation) so we get:

$\sigma(\Omega) = \frac{m^2}{4\pi^2}\left|\tilde{U}(q)\right|^2 = \frac{4m^2U_0^2}{\left(q^2+\frac{1}{a^2}\right)^2}$

now we choose $\theta$ to be the angle between $\vec{k_0}$ and $\vec{k_\Omega}$.

since $k_E = \left|\vec{k_\Omega}\right| = \left|\vec{k_0}\right|$ we have $q = 2k_E sin\left(\frac{\theta}{2}\right)$

now $\sigma(\Omega) = \frac{4m^2U_0^2}{\left(\frac{1}{a^2}+4k_E^2 sin^2\left(\frac{\theta}{2}\right)\right)^2}$

By taking the limit $a \to \infty$ the cross section of the Yukawa potential approaches the Rutherford scattering cross section:

$\sigma(\Omega)\left[a \to \infty\right] = \frac{4m^2U_0^2}{\left(4k_E^2 sin^2\left(\frac{\theta}{2}\right)\right)^2}$

Now, let's calculate the total cross section for the Yukawa potential:

$\sigma_{total} = \int\int \sigma(\Omega)d\Omega =$

$\sigma_{total} = \int_0^{2\pi} d\varphi \int_0^{\pi} sin\theta d\theta \cdot \sigma(\theta) = 2\pi \int_0^{\pi} \sigma(\theta) sin\theta d\theta$

It will be easier to use the expression for q :

$q = 2k_E sin\left(\frac{\theta}{2}\right) \;\Rightarrow dq = k_E cos\left(\frac{\theta}{2}\right) d\theta \cdot \frac{q}{q} = \frac{k_E^2}{q} sin\theta d\theta \quad \Rightarrow sin\theta d\theta = \frac{q \cdot dq}{k_E^2}$

$\sigma_{total} = \frac{4m^2U_0^2}{k_E^2} 2\pi \int_0^{2k_E} dq \cdot q \frac{1}{\left(q^2+\frac{1}{a^2}\right)^2}$

$z = q^2 \quad dz = 2q \cdot dq$

$\Rightarrow \sigma_{total} = \frac{4m^2U_0^2\pi}{k_E^2} \int_0^{4k_E^2} \frac{dz}{\left(\frac{1}{a^2}+z\right)^2} = \frac{4m^2U_0^2\pi}{k_E^2} \frac{-1}{\frac{1}{a^2}+z}\Big|_0^{4k_E^2} =$

$\frac{4m^2U_0^2\pi}{k_E^2}\left(a^2 - \frac{a^2}{1+4a^2k_E^2}\right) = 4m^2U_0^2\pi \frac{4a^4}{1+4a^2k_E^2}$

$\Rightarrow \sigma_{total} = \frac{16a^4m^2U_0^2\pi}{1+4a^2k_E^2}$