Ex 785: Calculation of the cross section for the Gauss potential

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The problem :
Find the differential cross section and the total cross section by the Born approximation for gauss potential : $U(r) = U_0 \mathrm{e}^{-\frac{(r/a)^2}{2}}$. Define the conditions for the using of the Born approximation .

The solution:
From the Fermi's Golden Rule we get a formula for the differential cross section called the Born approximation :

$$\sigma(\Omega) = (\frac{M}{2\pi})^2 |\tilde{U}(q)|^2$$

where $\tilde{U}(q)$ is the Fourier Transform of the scattering potential $U(r)$.

Lets calculate the $\tilde{U}(q)$ for the gauss potential :

$$\tilde{U}(q) = FT(U(r)) = \int\int\int U(r)\mathrm{e}^{-iqr}d^3r = \int\int\int U_0\mathrm{e}^{-r^2/2a^2}\mathrm{e}^{-iqr}d^3r =$$

$$= \int\int\int U_0\mathrm{e}^{-r^2/2a^2}\mathrm{e}^{-iqRcos\theta}d\phi d(cos\theta)R^2dR = 4\pi\int_0^\infty U_0\mathrm{e}^{-R/a}sinc(qR)R^2dR =$$

$$= 4\pi\frac{U_0}{q}\int_0^\infty R\mathrm{e}^{-R^2/2a^2}sin(qR)dR = \frac{2\pi U_0}{qi}\int_0^\infty R\mathrm{e}^{-R^2/2a^2}(\mathrm{e}^{iqR} - \mathrm{e}^{-iqR})dR$$

Lets use a new markings:

$$-\frac{R^2}{2a^2} \pm iqR = -\frac{1}{2a^2}(R \pm iqa^2)^2 - \frac{q^2a^2}{2}$$

$$l = R \pm iqa^2$$

$$dl = dR$$

$$FT[U(r)] = \frac{2\pi U_0}{iq}\mathrm{e}^{-\frac{q^2a^2}{2}}\int_0^\infty \mathrm{e}^{-\frac{l^2}{2a^2}}(l - iqa^2) - \mathrm{e}^{-\frac{l^2}{2a^2}}(l + iqa^2)dl =$$

$$= -4\pi U_0 a^2 \mathrm{e}^{-\frac{q^2a^2}{2}} \int_0^\infty \mathrm{e}^{-\frac{l^2}{2a^2}} dl = -U_0(a\sqrt{2\pi})^3 \mathrm{e}^{-\frac{q^2a^2}{2}}$$

So , the differential cross section is :

$$\sigma(\Omega) = 2\pi a^6 m^2 U_0^2 \mathrm{e}^{-q^2a^2}$$

Now , lets calculate the total cross section for the Gauss potential :

$$\sigma_{total} = \int\int \sigma(\Omega) d\Omega = (\frac{1}{2\pi})^2 (\frac{k_E}{v_E})^2 2\pi \int_{-1}^{1} |FT[U(q)]|^2 d(cos\theta_\Omega)$$

but because of $q = 2k_E sin(\theta_\Omega/2)$, we get :

$$\sigma_{total} = \frac{1}{2\pi v_E^2} \int_0^{2k_E} |FT[U(r)]|^2 q dq$$

Lets use the $FT[U(r)]$ that we found for given potential :

$$\sigma_{total} = \frac{1}{2\pi v_E^2} \int_0^{2k_E} | - U_0(a\sqrt{2\pi})^3 \mathrm{e}^{-\frac{q^2a^2}{2}} |^2 q dq =$$

$$\frac{U_0^2}{4\pi v_E^2} (2\pi a^2)^3 \int_0^{2k_E} \mathrm{e}^{-q^2a^2} dq^2 = -\frac{2\pi^2 a^4 U_0^2}{v_E^2} \left[\mathrm{e}^{-q^2a^2}\right]_0^{2k_E}$$

we get the finel result :

$$\sigma_{total} = \frac{2\pi^2 a^4 U_0^2}{v_E^2} \left[1 - \mathrm{e}^{-2k_E^2 a^2}\right]$$

the condition for the using of the Born approximation is derived from F.G.R (weak potential), and so we demand that $U << \Delta$ where Δ is the spacing of the "on site energies" and U is the pertubation.