Ex7684: Dot - Ring System

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The problem:

A particle of mass M is confined to a site. The undisturbed bond energy is ϵ_0.
The site is coupled to a large ring of length L at one point, so that $\langle \epsilon_0 | H | \Psi \rangle = \alpha \Psi(x_0)$.
In the first question assume that the energy ϵ_0 is underneath the potential floor of the ring.

(1) Calculate the correction Δ to the bond energy, by using second order perturbation theory.
(2) Repeat the first question under the assumption that ϵ_0 is above the potential floor.
(3) In the last case, calculate the decay constant Γ of the bound state, by using Fermi's golden rule.
(4) Find a domain of values for ϵ_0, in which the previous results are unreliable.
(5) The bond energy from the first question could be found accurately, by solving a third degree equation. Write the equation.

Guidance:

- It would be convenient to work in the momentum basis $|k\rangle$ of the ring.
- Assume that the ring is very big, so that the discrete sum can be approximated by an integral.
- Write the final answer using the constants α, M, ϵ_0.
- If it helps it is allowed, and advisable to use the notation: $v_0 = \left(\frac{2\epsilon_0}{M}\right)^{\frac{1}{2}}$.

The solution:

For clarity reasons let us denote by Ψ a general linear combination of the momentum states, and by ψ the general wave function.

(1) The system is a combination of two systems which would have been separate unless there was a coupling. The coupling exists only due to the disturbance. Therefore the general wave function for the undisturbed system is:

$$|\psi\rangle = a_1 |\epsilon_0\rangle + \sum_x \Psi(x) |x\rangle$$

The sum represnets all of the states which "live" on the ring, and the standalone state is the one other state.
Since the states on the ring are known, the previous statement can be refined by using the eigenstates of a ring, because this is a regular ring ignoring the coupling:

$$|\psi\rangle = a_1 |\epsilon_0\rangle + \frac{1}{\sqrt{L}} \sum_x e^{ik_n x} |x\rangle$$

with:

$$k_n = \frac{2\pi}{L} \times n, \qquad n = 0, \pm 1, \pm 2, ...$$

Now that the undisturbed Hamiltonian is clear, lets evaluate the disturbance.
Since the coupling occures in a specific site in the position basis, all of the momentum states are coupled to the site outside the ring. Let us choose the x-axis origin so that the coupling is at $x = 0$, thus:

$$\langle \epsilon_0 | H | \Psi_k \rangle = \langle \epsilon_0 | H \Psi^k(0) | 0 \rangle = \langle \epsilon_0 | H \frac{e^{ik_n \cdot 0}}{\sqrt{L}} | 0 \rangle = \frac{\alpha}{\sqrt{L}}$$

Therefore the coupling of each of the momentum states to standalone state is:

$$V_{n,\epsilon_0} = \frac{\alpha}{\sqrt{L}}$$

Now by using second order perturbation theory approximation, and remembering $\epsilon_0 < 0$:

$$\Delta = \sum_{k=-\infty}^{\infty} \frac{|V_{k,\epsilon_0}|^2}{-|\epsilon_0| - \epsilon_k} = -\frac{2M\alpha^2}{L} \sum_{k=-\infty}^{\infty} \frac{1}{k_0^2 + k^2} = -\frac{2M\alpha^2}{L} \int_{-\infty}^{\infty} \frac{L \mathrm{d}k}{2\pi} \frac{1}{k_0^2 + k^2} = -\frac{M\alpha^2}{k_0} = -\frac{\alpha^2}{v_0}$$

The discrete sum turned into the integral since the ring is large enough.

(2) The analysis is the same, only now $\epsilon_0 > 0$:

$$\Delta = \sum_{k=-\infty}^{\infty} \frac{|V_{k,\epsilon_0}|^2}{\epsilon_0 - \epsilon_k} = \frac{2M\alpha^2}{L} \sum_{k=-\infty}^{\infty} \frac{1}{k_0^2 - k^2} = \frac{2M\alpha^2}{L} \int_{-\infty}^{\infty} \frac{L \mathrm{d}k}{2\pi} \frac{1}{k_0^2 - k^2} = 0$$

The last integral can be evaluated by using partial fractions, one should remember the absolute value in the natural lograthim:

$$\int \frac{1}{x} \mathrm{d}x = \ln|x| + c$$

(3) In order to avoid confusion let us use δ for mean level spacing and not the usual Δ, because Δ is already taken for correction of the energy due to pertubation.
That being said, dispersion relation can be used in order to find the mean level spacing by:

$$\mathrm{d}E = v_E \mathrm{d}p \quad \Rightarrow \quad \delta = v_0 \frac{\pi}{L}$$

This is particularly useful since an assumption is made that the decay into the continuum occurs at the energy of the standalone state.
Now according to Fermi's golden rule:

$$\Gamma = \frac{2\pi}{\delta} |V_{n,\epsilon_0}|^2 = \frac{2\alpha^2}{v_0}$$

(4) In the first case (negative bond energy) in order for pertubation theory to be valid one must require:

$$\Delta << \epsilon_0$$

In the second question, as seen in the lecture notes for Fermi's golden rule to be valid one must assume: $\Gamma t << 1$
Now the question that must be answered is which time is relevant in the previous statement.
Let us define: $\tau \equiv \frac{1}{\epsilon_0}$
This characteristic time is the period of the oscillation of the standalone state. Clearly unless $\Gamma\tau << 1$, oscillations between this state and the continuum won't be seen, as seen in Fermi's golden rule.
Hence the requirement will be:

$$\Gamma\tau << 1 \quad \Rightarrow \quad \Gamma << \epsilon_0$$

(5) As seen in the Lec. notes in 43.2 the Schrödinger equation can be written as follows:

$$\epsilon_0|\epsilon_0\rangle + \frac{\alpha}{\sqrt{L}}\sum_k \Psi_k = E|\epsilon_0\rangle$$

$$\epsilon_k\Psi_k + \frac{\alpha}{\sqrt{L}}|\epsilon_0\rangle = E\Psi_k, \qquad k = 1, 2, 3, ...$$

the k-th equation may be written also as:

$$\Psi_k \quad = \quad \frac{\frac{\alpha}{\sqrt{L}}}{E_k - \epsilon_k}|\epsilon_0\rangle$$

And now by substituting this expression into the 0-th equation:

$$\frac{\alpha^2}{L}\sum_k \frac{1}{E - \epsilon_k} \quad = \quad E - \epsilon_0$$

This sum has been calculated in the first question, and for an arbitrary energy E:

$$-\frac{\alpha^2}{v_E} = E - \epsilon_0$$

By writing explicitly v_E, it is easy to find the eqution:

$$M\alpha^2 = 2E \cdot (E - \epsilon_0)^2$$

where $v_E = \sqrt{\frac{2E}{M}}$