Ex7680: Gamow decay from a ring

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The problem:

A particle with mass M in a device of a length L_0.
You may consider this device as "potential well" with boundaries at 0 and L_0 $(0 < x < L_0)$ with boundaries conditions 0.
The particle is prepared in one of the energy levels $E_n : n = 1, 2, 3, 4....$
The particles speed is defined by the expression $v_n = (2E_n/M)^{1/2}$.
The particle can escape to k states of a long one dimensional conductor.
We will treat the contact point as a delta function $V(r) = u\delta(r)$.
The matrix elements of the intersection are $V_{nk} = -\frac{1}{4M^2u}[\partial\psi^{(n)}][\partial\psi^{(k)}]$,
When ∂ refer to the differential in the radial direction (the r coordinate is pointing for the junction outward).
The transmission coefficients of the contact point are $g_1, g_2 << 1$.
Express your answers in terms of (L_0, M, v_n, g_1, g_2).

(1) Preparation question: write what are the energy levels E_n and the eigenstates $\Psi(n)$.
(2) Write an exact expression for the matrix elements V_{nk} between the contraption and the conductor states.
(3) Write an exact expression for the density of the energy states $\frac{dN}{dE}$ inside the long conductor.
(4) Find the Gamow decay constant Γ_n of each of the n states.
(5) Generalize the formula to a case where there is a magnetic flux through the ring. Assume the particle's charge is e.

Guidance and clues:
In (2) assume $g_2 = 0$. furthermore assume $E_k \approx E_n = \frac{1}{2}Mv_n^2$. We can assume the length of the long conductor is finite L but it should vanish in the answer in (4). The formula for a very big delta barrier is $g \approx (\frac{v_n}{u})^2$. (5) it required only physical understanding and not algebraic derivations.

The solution:

(1)
since the device is treated as one dimensional potential well $(0 < x < L_0)$, the particle states are of a particle in a one dimensional box. For a non-periodical boundary conditions $k_n = \frac{\pi n}{L_0}$. The energy levels are: $E_n = \frac{k_n^2}{2M} = \frac{\pi^2 n^2}{2ML_0}$ for the device and $E_k = \frac{k_k^2}{2M} = \frac{\pi^2 k^2}{2ML}$ for the long conductor. The eigenstates are:

$$\Psi^{(n)}(x) = \sqrt{\frac{2}{L_0}} sin(k_n x) = \sqrt{\frac{2}{L_0}} sin(\frac{\pi n x}{L_0})$$

for the long conductor, substitute L_0 by L:

$$\Psi^{(k)}(x) = \sqrt{\frac{2}{L}} sin(k_k x) = \sqrt{\frac{2}{L}} sin(\frac{\pi k x}{L_0})$$

(2)
We assume $g_2 = 0$
We know the matrix elements are $V_{nk} = -\frac{1}{4M^2u}[\partial\psi^{(n)}][\partial\psi^{(k)}]$
and under the assumptions $\left\{\frac{k_n}{M} = v_n\right\}, \{E_n \approx E_k\}, \left\{\sqrt{\frac{v_n}{u}} = g_1\right\}$ we will get

$$V_{nk} = -\frac{1}{4M^2u}\sqrt{\frac{2}{L_0}}k_n cos(0)\sqrt{\frac{2}{L}}k_k cos(0) \approx \frac{-\sqrt{g_1}v_n}{2\sqrt{LL_0}}$$

(3)
This is always true that $dE = v_k dp$
and we will get that:
density of energy states = the particle speed $\times$ quantization of momentum

$$\Delta = v_n\frac{\pi}{L}$$

and density states = $\frac{1}{\Delta}$

(4)

$g_2 \neq 0$

The matrix elements are :
$|V_{nk}| = |V_{nk}^{(1)} + V_{nk}^{(2)}|$
where $x = 0$ at $V_{nk}^{(1)}$, and $x = L_0$ at $V_{nk}^{(2)}$
we derive $V_{nk}^{(2)}$ at the direction the particle leaves the intersection which is opposite to the x direction.
so the matrix elements for $V_{nk}^{(2)}$ are:

$$V_{nk} = \frac{-\sqrt{g_2}v_n}{2\sqrt{LL_0}}cos(\pi n)(-1)cos(0)$$

we will get:

$$|V_{nk} = |\frac{-\sqrt{g_1}v_n}{2\sqrt{LL_0}} \pm \frac{-\sqrt{g_2}v_n}{2\sqrt{LL_0}}| = \frac{v_n^2}{4LL_0}|\sqrt{g_1} \pm \sqrt{g_2}|$$

the decay constant is defined as $\Gamma_n = \frac{2\pi}{\Delta}|V_n k|^2$ and we will get:

$$\Gamma_n = \frac{v_n}{2L_0}|\sqrt{g_1} \pm \sqrt{g_2}|^2$$

(5)

The magnetic flux is added to the equation between the edge of the device to
the long conductor, because it is the phase it collects while moving on the device.
so the matrix element will be $V_{nk}^{(2)} \to V_{nk}^{(2)}e^{ie\Phi}$

$$\Gamma_n = \frac{v_n}{2L_0}|\sqrt{g_1} \pm e^{ie\Phi}\sqrt{g_2}|^2$$