E738: Dynamics In Two Sites System, With Time driven perturbation

Submitted by: Oded Perry

The problem:

A system Can be in a two states with no jumping probablity between them
The energy that correspond to this states is diffrent from one another
the Hamiltonian is:

$$H = \begin{pmatrix} E_1 & \gamma e^{i\omega_0 t} \\ \gamma e^{-i\omega_0 t} & E_2 \end{pmatrix}$$

At t=0 the system is in ground state , find:

1. the probability to be in ground state,and in the second state, in the first order perturbation theory.
2. The probability to be in the ground state and in the second state excplicity.

The solution:

1. We can Write the Hemilitonian like this :

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} + \gamma e^{-i\omega_0 t} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \gamma e^{i\omega_0 t} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

assuming that $\omega > 0$ and $\omega_0 > 0$ and since we are looking for the probability to jump from one to two we take into considration only the $\gamma e^{-i\omega_0 t}$,We will write W like this

$$W = \gamma \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

now we can use the formula :

$$C_n(t) = -iW_{nm}FT[f(t)]$$

the probability to make a transtion is:

$$p(t) = p_t(2|1) = |C_n(t)|^2$$

And the probability to stay

$P(t) = P_t(1|1) = 1 - p_t(2|1)$

So we get

$$|C_n(t)|^2 = \gamma^2 FT[e^{-i\omega_0 t}]^2 = |\gamma|^2 |\frac{1 - e^{i(\omega-\omega_0)t}}{\omega - \omega_0}|^2 = \gamma^2 t^2 \text{sinc}^2\left(\frac{\omega - \omega_0}{2} t\right)$$

Where $\omega = E_2 - E_1$.

2. For simplicity we will define:

$$\omega = E_2 - E_1$$
$$\Omega = \omega - \omega_0$$

then the coefficient Equation are:

1. $\frac{\partial \mathrm{C}_1}{\partial \mathrm{t}} = -i\gamma C_2 e^{-i\Omega t}$
2. $\frac{\partial \mathrm{C}_2}{\partial \mathrm{t}} = -i\gamma C_1 e^{i\Omega t}$

Solving for C_1 we get:

$$C_1 = -\frac{i}{\gamma}\frac{\partial \mathrm{C}_2}{\partial \mathrm{t}} e^{-i\Omega t}$$

differentiating this equation and plugin it into equation number one. we get :

$$\frac{i}{\gamma}\left(\frac{\partial^2 \mathrm{C}_2}{\partial t} e^{-i\Omega t} - i\Omega \frac{\partial \mathrm{C}_2}{\partial \mathrm{t}} e^{-i\Omega t}\right) = -i\gamma \mathrm{C}_2 e^{-i\Omega t}$$

$$\frac{\partial^2 \mathrm{C}_2}{\partial t} - i\Omega \frac{\partial \mathrm{C}_2}{\partial \mathrm{t}} + \gamma^2 \mathrm{C}_2 = 0$$

This is a second order linear differential equation (SOLDE):

$$X^2 - i\Omega X + \gamma^2 = 0$$

$$X_{12} = \frac{i\Omega \pm \sqrt{-\Omega^2 - (4\gamma)^2}}{2} = i\frac{\Omega \pm \sqrt{\Omega^2 + (4\gamma)^2}}{2}$$

Hence the solution is :

$$\mathrm{C}_2 = Ae^{x_1 t} + Be^{x_2 t}$$

Using the intial condtions :

$$C_1(0) = 1$$
$$C_2(0) = 0$$
$$A + B = 0 \Rightarrow A = -B$$

We get :

$$C_2 = A(e^{x_1 t} - e^{x_2 t}) = Ae^{\frac{i\Omega t}{2}}\left(e^{i\frac{\sqrt{\Omega^2 + (2\gamma)^2}}{2}t} - e^{-i\frac{\sqrt{\Omega^2+(2\gamma)^2}}{2}t}\right) = 2iAe^{\frac{i\Omega t}{2}} \sin\left(\frac{\sqrt{\Omega^2 + (2\gamma)^2}}{2}t\right)$$

$$\frac{\partial C_2}{\partial t} = A(X_1 e^{x_1 t} - X_2 e^{x_2 t})$$

$$C_1 = \frac{iA}{\gamma}(X_1 e^{x_1 t - i\Omega t} - X_2 e^{x_2 t - i\Omega t})$$

$C_1(0) = 1$
$\frac{-A}{\gamma}(x_1 + x_2) = 1$
$A = \frac{-i\gamma}{\sqrt{\Omega^2+4\gamma^2}}$

thus the probabilty to make a transition:

$$p(t) = |\langle 2|\ \psi(t)\rangle|^2 = |C_2(t)|^2 = 4\,|A|^2 \sin^2\left(\frac{\sqrt{\Omega^2+(2\gamma)^2}}{2}t\right)$$

$$p(t) = \frac{4\gamma^2}{(\omega-\omega_0)^2+4(\gamma)^2}\sin^2\left(\frac{\sqrt{(\omega-\omega_0)^2+4\gamma^2}}{2}t\right) = \gamma^2 t^2 \mathrm{sinc}^2\left(\frac{\sqrt{(\omega-\omega_0)^2+4\gamma^2}}{2}t\right)$$

We can see that when γ is small we get the same solution as in the first order perturbation.