Ex7122: Transition between energy levels due to wall displacement

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The problem:

A particle with mass M is set in one dimensional box $x \in [0, a]$ with length a. At time $t = -\infty$ we prepare the particle at ground state $n = 1$ of the box. Displacement of the left wall is described by the step function $x = \epsilon(t)$.
The velocity in which we move the wall is $\dot{\epsilon} = \frac{\epsilon_0}{\sqrt{2\pi}\tau_0} exp[-\frac{1}{2}(\frac{t}{\tau_0})^2]$.
We measure the energy level of the particle at time $t = \infty$.

(1) Calculate the perturbation element $\dot{\epsilon}W_{n.1}$ for transitions of adiabatic states.
(2) Write the condition for the adiabatic approximation validity.
(3) Assume a small displacement $\epsilon_0 << a$ of the wall. Calculate the first order, using the perturbation theory (in the adiabatic basis), the probability $P(n)$ to find the particle in the exited level n.

We change the protocol as follows:
- First we move the left wall small displacement $\epsilon_0 << a$.
- We move the right wall in perfectly adiabatically formation to the point $x = b$.
- The displacement of the right wall is described by $x = a + v_0 t$ so the total time for the process is $t_0 = (b - a)/v_0$.
- Finally we return the left wall to it's initial place.

(4) Find the probability $P(n)$ for the new protocol.

Guidance: the perturbation element for infinitesimal displacement of the wall is
$< \varphi|V|\psi >= -(\frac{\epsilon}{2M})[\frac{\partial\varphi}{\partial x}][\frac{\partial\psi}{\partial x}]$.
The partial derivative are calculated in the point the wall is placed.
Express the results with the given parameters $(M, a, b, v_0, \epsilon_0, \tau_0)$.
Write the result of section 3 in the form $P(n) = |A(a)|^2$.
Express the result of section 4 with the function $A(\cdot)$ and the given parameters.

The solution:

(1) According to time dependent perturbation theory we can write, in the adiabatic basis, the perturbation element as follows:

$$W_{nm} = i < n|\frac{\partial m}{\partial x} >= \frac{-iV_{nm}}{E_n - E_m}$$

We can use the equation in the guidance:

$$V_{nm} = \frac{1}{2M}(\frac{\partial\varphi_n}{\partial x})|_{x=0}(\frac{\partial\psi_m}{\partial x})|_{x=0}$$

And for a particle in a one-dimensional box:

$$\varphi_n = \sqrt{\frac{2}{a}} \cdot sin(\frac{\pi n x}{a})$$

With the energies:

$$E_n = \frac{\pi^2 n^2}{2Ma^2}$$

Hence:

$$(\frac{\partial\varphi}{\partial x})|_{x=0} = \sqrt{\frac{2}{a}} \cdot \frac{\pi \cdot n}{a}$$

Therefore one gets:

$$V_{nm} = \frac{\pi^2 \cdot nm}{Ma^3}$$

$$W_{nm} = \frac{-i\pi^2 mn}{Ma^3 \cdot (E_n - E_m)}$$

And we get the result:

$$W_{n1} = -\frac{2i \cdot n}{a(n^2 - 1)}$$

(2) The adiabatic condition is given by:

$$\dot{X} << \frac{\Delta^2}{|V|}$$

One can observe that the energy level spacing in one dimensional box is given by:

$$\Delta = \frac{\pi \cdot v_E}{L}$$

For a wall displacement we have:

$$|V| = \frac{Mv_E^2}{L}$$

Then:

$$\dot{X} << \frac{\pi^2}{ML}$$

(3) The leading order of the transition probability from the ground state to state n is given by:

$$P_t(n|1) = |W_{n1}|^2 \times |FT[\dot{\epsilon}]|^2$$

Which results in:

$$P(n) = (\frac{2n}{a(n^2 - 1)})^2 \epsilon_0^2 e^{-(w_n \tau_0)^2}$$

Defining:

$$w_n \equiv E_n - E_1$$

(4) The leading order of the non-adiabatic transition probability is:

$$P_t(n|1) = |\int_0^t \frac{V_{n1}}{E_n - E_1} e^{-i(\Phi_n - \Phi_1)} \dot{X} dt|^2$$

Where:

$$\Phi_n(t) = \int_0^t E_n dt'$$

We notice that $\dot{X}$ isn't zero only when the right wall moves, so we can divide the integral into two integrals as follows:

$$P_t(n|1) = |\int_{part1} W_{n1} e^{-i(\Phi_n^{(1)})} \dot{X} dt' - \int_{part2} W_{n1} e^{-i(\Phi_n^{(2)})} \dot{X} dt'|^2$$

The velocity in the second part is in the opposite direction which results in a minus sign. Also, ω_n depends on X so the phase will be different for each integral. The phase integral limits will be from 0 to the current time and will be divided into 3 intervals: left wall movement, adiabatic right wall movement and left wall movement.
The expression for the frequency is:

$$\omega_n(X) = \frac{\pi^2(n^2-1)}{2MX^2}$$

We can see that for the left integral the phase will be:

$$\Phi_n^{(1)} = \int_0^t \omega_n(a) dt = \omega_n(a) t$$

The second phase includes the phase accumulated during the adiabatic process.

$$\Phi_n^{(2)} = \int_0^t \omega_n(a) dt + \int_{adiabatic} \omega_n(X) dt$$

Calculating the adiabatic part as follows:

$$\Phi_{adiabatic} = \int_a^b \frac{\pi^2(n^2-1)}{2MX^2} \frac{dt}{dX} dX = \frac{1}{v_0} \frac{\pi^2(n^2-1)}{2M} (\frac{1}{a} - \frac{1}{b})$$

Finally, the probability is:

$$P_t(n|1) = |A(a) - e^{-i\Phi_{adiabatic}} A(b)|^2$$

Defining:

$$A(x) \equiv 2\frac{\epsilon_0}{x} (\frac{n}{n^2-1}) e^{-\frac{1}{2}(\tau_0 \omega_n(x))^2}$$