E712: Transitions between energy levels due to wall movment

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The problem:

A particle with mass M in 1D box $x \in [0, a]$ with length a.
At time $t = -\infty$ the particle is in ground state.
The movment of the wall is $x = -\epsilon(t)$.
The speed in which the wall is moving is $\dot{\epsilon} = \frac{\epsilon_0}{\sqrt{2\pi}\tau_0} \exp[-\frac{1}{2}(\frac{t}{\tau_0})^2]$.
You may assume that the displacement of the wall is small so that the perturbation is

(*)$< \varphi|V|\psi >= \mp(\frac{\epsilon}{2M})[\frac{\partial\varphi}{\partial x}][\frac{\partial\psi}{\partial x}]$.

- The derivatives are calculated at the point where the wall is located.
- The sign +/- refer to the case of enlargement/reduction (respectively) of the box's length.

(1) Find the probability $P(n)$,at time $t = \infty$, to find the particle at an excited energy level n for abrupt displacement.
(2) Using first order Perturbation theory, find the probability $P(n)$ for finite displacement.
(3) Answer paragraph (2) only this time both of the walls are displaced outward in the next fashion $[-\epsilon(t),\, a + \epsilon(t)]$.
(4) Define two different conditions that can ensure the Validity of the calculation of first order Perturbation theory.
-Small displacement of the wall (small ϵ_0) or a slow movment of the wall (large τ_0).
(5) Write the condition needed for the process to be adiabatic and note if the adiabatic area is

contained within the validity area of the Perturbation theory.

The solution:

(1) For 1D potential well the energy levels and wave function are:

$E_n = \frac{\pi^2}{2Ma^2}n^2$, $\psi_n = \sqrt{\frac{2}{a}}sin(\frac{\pi nx}{a})$.

The probability of a particle to pass from energy level m to evergy level n is:

$P(n|m) = \frac{|V_{nm}|^2}{(E_n - E_m)^2}$, for $n \neq 1$.

using equation (*) we see that :

$V_{n1} = \frac{-\epsilon(t)\pi^2}{Ma^3}n$.

the overall displecment of the wall is

$$\int\limits_{-\infty}^{\infty} \dot{\epsilon}(t)dt = \frac{\epsilon_0}{\sqrt{2\pi}\tau_0} \int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}(\frac{t}{\tau_0})^2} dt = \epsilon_0$$

thus at $t = \infty$

$V_{n1} = \frac{-\epsilon_0\pi^2}{Ma^3}n$.

In our case

$P(n|1) = \frac{|V_{n1}|^2}{(E_n - E_1)^2}$

$E_n - E_1 = \frac{\pi^2}{2Ma^2}(n^2 - 1)$,

Therefore

$P_0(n) = P(n|1) = 4\frac{\epsilon_0}{a^2}(\frac{n}{n^2-1})^2$ for $n \neq 1$.

(2) using the formula for the transition probability:

$P(n) = |w_{nm}|^2|FT(f(t))|^2$

$|w_{n1}| = \frac{\pi^2}{Ma^3}n$

$$FT(f(t)) = FT(\epsilon(t)) = \int\limits_{-\infty}^{\infty} \epsilon(t)e^{-iw_nt}dt = \frac{\epsilon(t)e^{-iw_nt}}{iw_nt} |_{-\infty}^{\infty} + \frac{1}{iw_n} \int\limits_{-\infty}^{\infty} \dot{\epsilon}(t)^{-iw_nt}$$

where :

w$_n = E_n - E_1$

the first term vanishes ($\epsilon(-\infty) = 0$) and from sloving the integral by parts we get:

$$\frac{\epsilon_0}{\sqrt{2\pi}\tau_0} \int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}(\frac{t}{\tau_0})^2} e^{-iw_nt}dt = \frac{\epsilon_0}{\sqrt{2\pi}\tau_0} e^{-\frac{(\tau_0 w_n)^2}{2}} \int\limits_{-\infty}^{\infty} e^{-\frac{(t+iw_n\tau_0^2)^2}{2\tau_0^2}} dt = \epsilon_0 e^{-\frac{(\tau_0 w_n)^2}{2}}$$

so finaly we get:

$FT(\epsilon(t)) = \frac{1}{iw_n}\epsilon_0 e^{-\frac{(\tau_0 w_n)^2}{2}}$

$P(n) = \frac{\epsilon_0^2 |w_{n1}|^2}{w_n^2} e^{-(\tau_0 w_n)^2} = P_0(n)e^{-(\tau_0 w_n)^2}$

(3) using (*) on both walls we get:

$V_{n1} = \frac{-\epsilon(t)\pi^2}{Ma^3}n + \frac{-\epsilon(t)\pi^2}{Ma^3}n(-1)^n = -2\epsilon(t)\frac{\pi^2}{Ma^3}n$

n=3,5,7...

from here it is easy to see that by following the same prosses as in the last section we will get:

$P`(n) = 4P(n)$,for n=3,5,7...

$P`(n) = 0$, for n=2,4,6...

(4) in order for the perturbation theory to be valid P(n) has to be small ,from here we get the conditions:

$\epsilon_0 << a$

$\tau_0 w_n >> 1 \Rightarrow \tau_0 >> ma^2$

(5) the adiabatic condition is :

$\dot{\epsilon} << \frac{\triangle^2}{|w|}$

$|w| = \frac{1}{ma}K_n K_m = \frac{m}{a}V_E^2$

$\triangle = V_E \triangle p = \frac{\pi}{a}V_E$, $(\pi \simeq 1)$

where we have used the aproximations:

$K_n \simeq K_m \simeq K_E = mV_E$

$\Rightarrow \dot{\epsilon} << \frac{1}{ma}$
and substituting the expression for $\dot{\epsilon}(t)$ we get:

$\epsilon_0 << \frac{1}{ma}\tau_0$

which is contained within the validity area of the Perturbation theory