E7110: Sudden movement of a wall

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The problem:

A particle is placed in potential well $0 < x < L$, in his ground state.
Suddenly, the wall is moved, so the wide of the well is $-L < x < L$.
Find the probability $P(n)$ to find the particle in energy level n.

The Solution:

The particle was placed in the ground state of a potential well $0 < x < L$, therefore his wave function is:

$$\psi_0 = \frac{1}{\sqrt{\frac{L}{2}}} \sin\left(\frac{\pi}{L}x\right) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}x\right) \qquad for \;\; 0 < x < L$$

After the sudden movement of the wall of the well, the particle remains in the same wave function, but the basic functions had changed.

$$\psi_n = \begin{cases} \frac{1}{\sqrt{L}} \sin\left(\frac{\pi n}{2L}x\right) & for\ n - even \\ \frac{1}{\sqrt{L}} \cos\left(\frac{\pi n}{2L}x\right) & for\ n - odd \end{cases}$$

$$\psi_0 = \sum_n C_n \cdot \psi_n$$

$$C_n = <\psi_n|\psi_0> = \int_{-L}^{L} \psi_n^* \cdot \psi_0 dx = \int_0^L \psi_n^* \cdot \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}x\right) dx$$

for n - even

$$C_n = \int_0^L \frac{1}{\sqrt{L}} \sin\left(\frac{\pi n}{2L}x\right) \cdot \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}x\right) dx = \frac{\sqrt{2}}{L} \int_0^L \sin\left(\frac{\pi n}{2L}x\right) \cdot \sin\left(\frac{\pi}{L}x\right) dx =$$

$$= \left[\frac{\sqrt{2}}{L} \frac{\frac{\pi(n-2)}{2L} \sin\left(\frac{\pi(n+2)}{2L}x\right) + \frac{\pi(n+2)}{2L} \sin\left(\frac{\pi(n-2)}{2L}x\right)}{2 \cdot \left(\left(\frac{\pi}{L}\right)^2 - \left(\frac{\pi\ n}{2L}\right)^2\right)}\right]_{x=0}^{x=L} =$$

$$= \frac{\sqrt{2}}{L} \frac{\frac{\pi(n-2)}{2L} \sin\left(\frac{\pi(n+2)}{2}\right) + \frac{\pi(n+2)}{2L} \sin\left(\frac{\pi(n-2)}{2}\right)}{2 \cdot \left(\left(\frac{\pi}{L}\right)^2 - \left(\frac{\pi\ n}{2L}\right)^2\right)}$$

n is an even number, therfore, $\frac{n\pm 2}{2} = m$, where m is integer.

$$\sin(m\pi) = 0$$

therefore,

$$C_n = 0, \quad \text{for n - even.}$$

for n - odd

$$C_n = \int\limits_0^L \frac{1}{\sqrt{L}} \cos\left(\frac{\pi n}{2L} x\right) \cdot \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L} x\right) dx = \frac{\sqrt{2}}{L} \int\limits_0^L \cos\left(\frac{\pi n}{2L} x\right) \cdot \sin\left(\frac{\pi}{L} x\right) dx =$$

$$= \left[-\frac{\sqrt{2}}{L} \frac{\frac{\pi(2-n)}{2L} \cos\left(\frac{\pi(2+n)}{2L} x\right) + \frac{\pi(2+n)}{2L} \cos\left(\frac{\pi(2-n)}{2L} x\right)}{2 \cdot \left(\left(\frac{\pi}{L}\right)^2 - \left(\frac{\pi\ n}{2L}\right)^2\right)} \right]_{x=0}^{x=L} =$$

$$= -\frac{\sqrt{2}}{L} \frac{\frac{\pi(2-n)}{2L} \cos\left(\frac{\pi(2+n)}{2}\right) + \frac{\pi(2+n)}{2L} \cos\left(\frac{\pi(2-n)}{2}\right)}{2 \cdot \left(\left(\frac{\pi}{L}\right)^2 - \left(\frac{\pi\ n}{2L}\right)^2\right)} + \frac{\sqrt{2}}{L} \frac{\frac{\pi(2-n)}{2L} + \frac{\pi(2+n)}{2L}}{2 \cdot \left(\left(\frac{\pi}{L}\right)^2 - \left(\frac{\pi\ n}{2L}\right)^2\right)}$$

n is an odd number, therefore, $\frac{n \pm 2}{2} = m + \frac{1}{2}$, where m is integer.

$$\cos\left(\left(m + \tfrac{1}{2}\right)\pi\right) = 0$$

therefore,

$$C_n = \frac{\sqrt{2}}{L} \frac{\frac{\pi(2-n)}{2L} + \frac{\pi(2+n)}{2L}}{2 \cdot \left(\left(\frac{\pi}{L}\right)^2 - \left(\frac{\pi\ n}{2L}\right)^2\right)} = \frac{4\sqrt{2}}{\pi(4-n^2)} \quad \text{for n - odd}$$

finally:

$$P(n) = |C_n|^2 = \begin{cases} 0 & for\ n - even \\ \frac{32}{\pi^2(4-n^2)^2} & for\ n - odd \end{cases}$$